EECS 20N: Structure and Interpretation of Signals and Systems QUIZ 2
Department of Electrical Engineering and Computer Sciences 17 October 2005
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name dès State-Space FIRST Name Alien

Lab Time Twilight

- (5 Points) Print your name and lab time in legible, block lettering in the appropriate spaces provided above.
- This quiz should take you up to 15 minutes to complete. You will be given at least 15 minutes—up to a maximum of 20 minutes—to work on the quiz.
- **This quiz is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the quiz. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct.
- **The quiz printout consists of pages numbered 1 through 6.** When you are prompted by the teaching staff to begin work, verify that your copy of the quiz is free of printing anomalies and contains all of the six numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your quiz. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you will receive full credit for the problem *only if* you justify your answer and explain your work clearly.
- We hope you do a *fantastic* job on this quiz!

Problem	Points	Your Score
Name	5	5
1	20	20
2	20	20
Total	**45**	45

Q2.1 (20 Points) For each system

$$F : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$$

described below, the input signal is denoted by x and the output signal by y. Determine which, if any, of the following properties each system possesses: **(I)** linearity, **(II)** time invariance, **(III)** causality, and **(IV)** memorylessness. Explain your reasoning clearly, but succinctly.

(a) [10 Points] $y(n) = x(n^2)$, $\forall n \in \mathbb{Z}$.

(I) F is Linear. Let x be a superposition of x_1 and x_2, i.e., $x(n) = \alpha x_1(n) + \alpha x_2(n)$, where $\alpha, \beta \in \mathbb{R}$ and $x_1, x_2 \in [\mathbb{Z} \to \mathbb{R}]$. Then $y(n) = x(n^2) = \alpha x_1(n^2) + \beta x_2(n^2) = \alpha y_1(n) + \beta y_2(n)$. Hence, $y = \alpha y_1 + \beta y_2$.

(II) F is not time invariant. Note that $y(n - n_0) = x((n - n_0)^2)$, where $n_0 \in \mathbb{Z}$ denotes an arbitrary integer shift. Let $\widehat{x}$ be defined such that $\widehat{x}(n) = x(n - n_0)$. Then $\widehat{y}(n) = \widehat{x}(n^2) = x(n^2 - n_0)$, which is generally not equal to $y(n - n_0)$.

(III) F is not causal. Note, for example, that $y(2) = x(4)$, so the system peeks ahead.

(IV) F is not memoryless. We can refer to the same counterexample used for disproving causality. Also, any system that is not causal cannot be memoryless.

(b) [10 Points] $y(n) = \cos(x(n))$, $\forall n \in \mathbb{Z}$.

(I) F is not linear. Let $\widehat{x} = \alpha x$, where $\alpha \in \mathbb{R}$ is an arbitrary scaling constant. Then $\widehat{y}(n) = \cos(\alpha x(n))$. However, $\alpha y(n) = \alpha \cos(x(n))$. Clearly, $\exists \alpha \in \mathbb{R}$ such that $\widehat{y} \neq \alpha y$. Alternatively, we can show that a zero-input (i.e., an input x such that $x(n) = 0, \forall n \in \mathbb{Z}$) produces a non-zero output; in fact, if $x = 0$, then $y(n) = 1, \forall n \in \mathbb{Z}$.

(II) F is time invariant. Note that $y(n - n_0) = \cos(x(n - n_0))$, where $n_0 \in \mathbb{Z}$ is an arbitrary shift. Define the signal $\widehat{x}$ such that $\widehat{x}(n) = x(n - n_0)$. The corresponding output $\widehat{y}$ is then characterized by: $\widehat{y}(n) = \cos(\widehat{x}(n)) = \cos(x(n - n_0))$, which equals $y(n - n_0)$.

(IV) F is memoryless. This is by inspection. The input-output relation $y(n) = \cos(x(n))$ is in the prototypical form of a memoryless system. (Refer to the definition given earlier in the course).

(III) F is causal. To determine any sample value of the output, the system does not "peek ahead" into the "future" values of the input. Alternatively, we showed that F is memoryless. Therefore, it must be causal as well.

NOTE: Every memoryless system is time invariant. Why? However, not every time-invariant system is memoryless. Can you think of a system that is time invariant but not memoryless?

Q2.2 (20 points) Consider a causal, discrete-time, SISO LTI system having the following $[A, B, C, D]$ state-space representation ($\forall n \geq 0$):

$$\underbrace{\begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix}}_{s(n+1)} = \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix}}_{s(n)} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} x(n)$$

$$y(n) = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} s(n) + \underbrace{1}_{D} x(n) .$$

The input signal, the output response, and the state response are $x : \mathbb{N}_0 \to \mathbb{R}$, $y : \mathbb{N}_0 \to \mathbb{R}$, and $s : \mathbb{N}_0 \to \mathbb{R}$, respectively.

(a) [10 Points] For this part only, suppose the input signal x is zero (i.e., $x(n) = 0,\ \forall n \geq 0$) and the initial state

$$s(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

Determine a simple expression for $s(n)$, the state of the system at time n ($\forall n \geq 1$).

This part is concerned with the zero-input state response. To see if a pattern emerges, write out the first few terms of the state $s(n)$:

$$s(1) = As(0) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad s(2) = As(1) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2^2 \\ 0 \end{bmatrix}$$

$$s(3) = As(2) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2^3 \\ 0 \end{bmatrix} \qquad s(n) = As(n-1) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2^{n-1} \\ 0 \end{bmatrix} .$$

Hence,

$$s(n) = \begin{bmatrix} 2^n \\ 0 \end{bmatrix} .$$

<u>NOTE</u>: If you tried to solve this part by computing $A^2, A^3, \ldots, A^n$, then you did far more work than the problem asked for. In fact, $A^n s(0)$ is rarely the best way to compute $s(n)$; usually, it is easier to proceed step-by-step, as shown above. In the last paragraph of p. 182 of the textbook, an analogous statement is made about the impulse response of an LTI system.

(b) [10 Points] For (i) and (ii) below, assume that the initial state is zero, i.e.,

$$s(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

(i) [7 Points] Determine a pair of input signal values $x(0)$ and $x(1)$ so that the state of the system at time $n = 2$ is given by:

$$s(2) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} .$$

Is your answer unique? If so, explain. If not, specify another pair of input sample values $x(0)$ and $x(1)$ that produces the same target state $s(2)$.

This part is concerned with the zero-state state response. Using the state update equation, we can express the state $s(n)$ in terms of the input signal values $x(0)$ and $x(1)$:

$$\begin{aligned} s(1) &= Bx(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(0) = \begin{bmatrix} x(0) \\ 0 \end{bmatrix} \\ s(2) &= Ax(1) + Bx(1) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x(0) \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(1) = \begin{bmatrix} 2x(0) + x(1) \\ 0 \end{bmatrix} . \end{aligned}$$

To determine appropriate input sample values $x(0)$ and $x(1)$, we set the expression derived above for $s(2)$ equal to the desired state $[3 \ \ 0]^{\mathsf{T}}$. This necessitates solving the underconstrained equation

$$2x(0) + x(1) = 3 ,$$

which has an infinity of solutions. Two exemplary solutions are $(x(0), x(1)) = (1, 1)$ and $(x(0), x(1)) = (2, -1)$.

(ii) [3 Points] Determine a pair of input signal values $x(0)$ and $x(1)$ so that

$$s(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

or explain (clearly, but succinctly) why no such $x(0)$ and $x(1)$ can be found.

Based on the expression for $s(2) = [s_1(2) \ \ s_2(2)]^{\mathsf{T}}$ found in part (i), it is clear that the input cannot control $s_2(2)$. Therefore, $s(2) = [0 \ \ 1]^{\mathsf{T}}$ is not a reachable state of the system and no solution exists for $x(0)$ and $x(1)$ in this case.