1 (a)

$$s(n+1) = As(n) + Bx(n) \qquad s(0) = \begin{bmatrix} s_1(0) \\ s_2(0) \end{bmatrix}$$

Zero-Input State Response $\rightarrow x(n) = 0,\ \forall n$

$$\begin{aligned}
s(n+1) &= As(n);\ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
s(1) &= As(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1(0) \\ s_2(0) \end{bmatrix} \\
&= \begin{bmatrix} s_1(0) + s_2(0) \\ s_2(0) \end{bmatrix} \\
s(2) &= As(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s_1(0) + s_2(0) \\ s_2(0) \end{bmatrix} \\
&= \begin{bmatrix} s_1(0) + 2s_2(0) \\ s_2(0) \end{bmatrix} \\
s(n) &= \begin{bmatrix} s_1(0) + ns_2(0) \\ s_2(0) \end{bmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
h(n) &= c^T A^{n-1} b,\ n \geq 1;\ h(0) = d = 0 \\
A^0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
A^1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
A^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\
A^n &= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \forall n \geq 0 \\
h(1) &= c^T A^0 b = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0] \\
h(2) &= c^T A^1 b = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1] \\
h(n) &= c^T A^{n-1} b = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & n-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [n-1]
\end{aligned}$$

(c)

$$
\begin{aligned}
s(0) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ x(n) = \delta(n-1) = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise} \end{cases} \\
\forall n \geq 0, &\quad y(n) = c^T A^n s(0) + \sum_{k=0}^{n} h(n-k)x(k) \\
y(0) &= c^T A^0 s(0) + h(0)x(0) = c^T A^0 s(0) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1] \\
y(1) &= c^T A^1 s(0) + \sum_{k=0}^{1} h(1-k)x(k) = c^T A^1 s(0) + h(0) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [0] \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [0] = [2] \\
y(n) &= c^T A^n s(0) + \sum_{k=0}^{n} h(1-k)x(k) = c^T A^n s(0) + h(n-1) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + h[n-1] \\
&= \begin{bmatrix} 1 & n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + h[n-1] = [1+n] + [n-2] = [2n-1]
\end{aligned}
$$

2

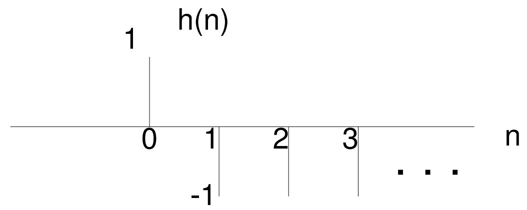
$$
\begin{aligned}
y(n) - y(n-1) &= x(n) - 2x(n-1) \\
y(n) &= x(n) - 2x(n-1) + y(n-1)
\end{aligned}
$$

(a)

$$
\begin{aligned}
s(n) &= \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} \\
s(n+1) &= \begin{bmatrix} y(n) \\ x(n) \end{bmatrix} = A \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + Bx(n) \\
&= \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} x(n) \\
y(n) &= C \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + dx(n) \\
&= \underbrace{\begin{bmatrix} 1 & -2 \end{bmatrix}}_{C} \begin{bmatrix} y(n-1) \\ x(n-1) \end{bmatrix} + \underbrace{1}_{d} x(n)
\end{aligned}
$$

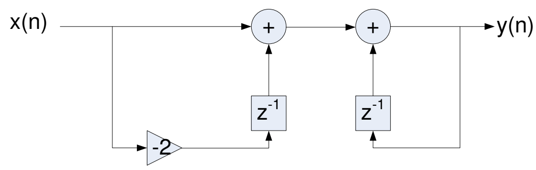
Zero State Impulse Response

$$
\begin{aligned}
s(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix};\ x(n) = \delta(n) \\
y(0) = d = 1 \qquad & s(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \\
y(1) = Cs(1) \qquad & s(1) = B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \\
y(2) = Cs(2) \qquad & s(2) = As(1) \\
y(3) = Cs(3) \qquad & s(3) = A^2 s(1) \\
\vdots \qquad & \vdots \\
y(n) = Cs(n) \qquad & s(n) = A^{n-1} s(1) \\
A^n &= \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\
A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\
h(0) &= 1 \\
h(1) &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 \\
h(2) &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -1 \\
&\vdots \\
h(n) &= -1
\end{aligned}
$$



(b)

$$y(n) = x(n) - 2x(n-1) + y(n-1)$$

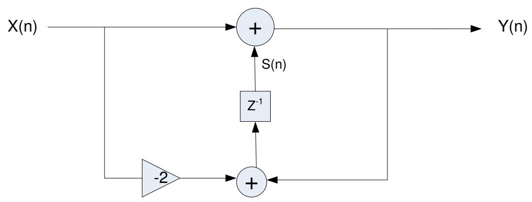


(c)

$$s(n) = -2x(n-1) + y(n-1)$$
$$s(n+1) = -2x(n) + y(n) = -x(n) + s(n)$$
$$A = 1, B = -1, C = 1, d = 1$$

Zero-State Impulse Response

$$\begin{array}{rl} s(0) = 0 & y(0) = 1 \\ s(1) = -1 & y(1) = -1 \\ s(2) = -1 & y(2) = -1 \\ \vdots & \vdots \end{array}$$



(d) Same as (a)

(e) Frequency response

$$\begin{aligned}
x(n) &= e^{j\omega n},\ y(n) = H(\omega)e^{j\omega n} \\
H(\omega)e^{j\omega n} &= e^{j\omega n} - 2e^{j\omega(n-1)} + H(\omega)e^{j\omega(n-1)} \\
H(\omega)(1-e^{-j\omega}) &= 1 - 2e^{-j\omega} \\
H(\omega) &= \frac{1-2e^{-j\omega}}{1-e^{-j\omega}}
\end{aligned}$$

Using impulse response:

$$\begin{aligned}
h(n) &= \delta(n) - u(n-1) \\
\delta(n) &\xrightarrow{DTFT} 1 \\
u(n) &\xrightarrow{DTFT} \frac{1}{1-e^{-j\omega}} \\
u(n-1) &\xrightarrow{DTFT} \frac{e^{-j\omega}}{1-e^{-j\omega}} \\
H(\omega) &= 1 - \frac{e^{-j\omega}}{1-e^{-j\omega}} = \frac{1-2e^{-j\omega}}{1-e^{-j\omega}}
\end{aligned}$$

3 (a)

$$s(n+1) = As(n) + Bx(n) \qquad s(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

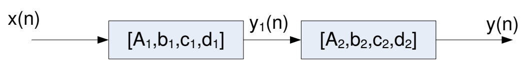
$$\begin{aligned}
s(2) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = As(1) + Bx(1) \\
s(1) &= As(0) + Bx(0) = A\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) + Bx(0) = Bx(0) \\
s(2) &= A(Bx(0)) + Bx(1) \longrightarrow \text{plug in } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} x(0)\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(1) \\
&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times (0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times (1) \\
x(0) &= [1] \\
x(1) &= [1]
\end{aligned}$$

(b)

$$s(0) = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$\begin{aligned}
s(1) &= As(0) + Bx(0) = A\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) + Bx(0) = Bx(0) \\
s(2) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = As(1) + Bx(1) \\
&= A\left(A\left(\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}\right) + Bx(0)\right) + Bx(1) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(0)\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(1) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} s_1 + s_2 \\ s_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x(0) \end{bmatrix}\right) + \begin{bmatrix} 0 \\ x(1) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} s_1 + s_2 \\ s_2 + x(0) \end{bmatrix}\right) + \begin{bmatrix} 0 \\ x(1) \end{bmatrix} \\
&= \begin{bmatrix} 2s_1 + s_2 + x(0) \\ s_1 + s_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x(1) \end{bmatrix} \\
&= \begin{bmatrix} 2s_1 + s_2 + x(0) \\ s_1 + s_2 + x(1) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
2s_1 + s_2 + x(0) &= 0 \\
s_1 + s_2 + x(1) &= 0 \\
x(0) &= -2s_1 - s_2 \\
x(1) &= -s_1 - s_2
\end{aligned}$$

4



$$
\begin{aligned}
y_1(n) &= c_1x(n) + d_1s_1(n) \\
y(n) &= c_2y_1(n) + d_2s_2(n) \\
&= c_2(c_1x(n) + d_1s_1(n)) + d_2s_2(n) \\
&= c_1c_2x(n) + c_2d_1s_1(n)d_2s_2(n) \\
s &= \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix}, C = c_1c_2, D = \begin{bmatrix} c_2d_1 & d_2 \end{bmatrix} \\
s_1(n+1) &= A_1s_1(n) + b_1x_2(n) \\
s_2(n+1) &= A_2s_2(n) + b_2y_1(n) \\
&= A_2s_2(n) + b_2[c_1x(n) + d_1s_1(n)] \\
&= c_1b_2x(n) + b_2d_1s_1(n) + A_2s_2(n) \\
\begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ (b_2d_1) & A_2 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \begin{bmatrix} b_1 \\ c_1b_2 \end{bmatrix} x(n) \\
A &= \begin{bmatrix} A_1 & 0 \\ b_2d_1 & A_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ c_1b_2 \end{bmatrix}
\end{aligned}
$$

5

$$
\begin{aligned}
x(n) = \delta(n) + \delta(n-2) \longrightarrow y(n) &= \delta(n) + \delta(n-1) + \delta(n-2) \\
&= x(n) + x(n-1) - x(n-3) + x(n-5) - x(n-7) \ldots \\
h(n) &= \delta(n) + \delta(n-1) - \delta(n-3) + \delta(n-5) - \ldots
\end{aligned}
$$

6

$$
x(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau
$$

(a)

$$
\begin{aligned}
h_1(t) &= \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases} \\
y_1(t) &= \int_0^{\infty} e^{-(t-\tau)}d\tau \quad t - \tau \geq 0 \\
&= e^{-t}\int_0^t e^t d\tau \\
&= e^{-t}e^{\tau}|_0^t \\
&= e^{-t}(e^t - 1) \\
&= (1 - e^{-t})
\end{aligned}
$$

(b)

$$
\begin{aligned}
h_2(t) &= \begin{cases} e^t & t < 0 \\ 0 & t \geq 0 \end{cases} \\
y_2(t) &= \int_0^\infty h(t-\tau)d\tau \qquad t-\tau < 0 \\
if \quad t < 0, \qquad y_2(t) &= \int_0^\infty e^t - \tau d\tau \\
&= -e^t e^{-\tau}|_0^\infty \\
&= -e^t(0-1) \\
&= e^t \\
if \quad t > 0, \qquad y_2(t) &= \int_t^\infty e^t - \tau d\tau \\
&= -e^t e^{-\tau}|_t^\infty \\
&= -e^t(0-e^{-t}) \\
&= 1 \\
y_2(t) &= \begin{cases} e^t & t \ll 0 \\ 1 & t > 0 \end{cases}
\end{aligned}
$$

(c)

$$
\begin{aligned}
h_3(t) &= \begin{cases} e^t & t < 0 \\ e^{-t} & t \geq 0 \end{cases} \\
y_3(t) &= \int_0^\infty h(t-\tau)d\tau \\
if \quad t < 0, \qquad \int_{-\infty}^t e^t dt &= e^t|_{-\infty}^t = e^t - 0 = e^t \\
if \quad t \geq 0 \qquad \int_{-\infty}^0 e^t dt + \int_0^t e^{-t} dt &= e^t|_{-\infty}^0 - e^{-t}|_0^t \\
&= 1 - (e^{-t} - 1) \\
&= 2 - e^{-t} \\
y_3(t) &= \begin{cases} 2 - e^{-t} & t \geq 0 \\ e^t & t < 0 \end{cases}
\end{aligned}
$$

7 (a) $x(t) = \cos(2\pi t) + \sin(3\pi t)$ Both components of $x(t)$ are periodic. $\therefore$ Period of x is the smallest p such that $2\pi p, 3\pi p$ are integer multiples of $2\pi \Rightarrow p = 2$ $\therefore$ x periodic with period 2.

$\Rightarrow$ Angular freq $\omega_0 = 2\pi/p = \pi$ rad/s
Using

$$\begin{aligned} \cos(2\pi t) &= \frac{e^{i2\pi t} + e^{-i2\pi t}}{2} \\ \sin(3\pi t) &= \frac{e^{i3\pi t} - e^{-i3\pi t}}{2i} \end{aligned}$$

we get the exponential Fourier Series representation of x as:

$$x(t) = \frac{i}{2}e^{-i3\pi t} + \frac{1}{2}e^{-i2\pi t} + \frac{1}{2}e^{i2\pi t} - \frac{i}{2}e^{i3\pi t}$$

Noting that $3\pi t = 3\omega_0 t, 2\pi t = 2\omega_0 t$, we get

$$x_m = \begin{cases} 1/2 & \text{when } m = \pm 3 \\ i/2 & \text{when } m = -3 \\ -i/2 & \text{when } m = 3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} z(t) &= \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases} \\ \text{CTFT} \quad z(\omega) &= \int_{-\infty}^{\infty} z(t)e^{-i\omega t}dt \\ &= \int_{0}^{\infty} e^{-t}e^{-i\omega t}dt \\ &= \int_{0}^{\infty} e-(1+i\omega)dt \\ &= \frac{e^{-(1+i\omega)t}}{-(1+i\omega)}|_0^\infty \\ &= \frac{1}{1+i\omega} \\ \therefore z(\omega) &= \frac{1}{1+i\omega} \end{aligned}$$

Using the frequency shifting property of CTFT, we get

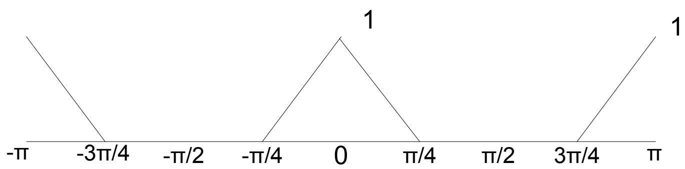
$$y(\omega) = z(\omega - \omega_0) = \frac{1}{1 + i(\omega - \omega_0)}$$

(c) (i) DTFT symmetric about $\omega = 0$
$\Rightarrow x(\omega) = x(-\omega)$
DTFT real $\Rightarrow x(\omega) = x^k(\omega)$ combining, we get $x(\omega) = x^k(\omega)$. Taking inverse DTFT, we get
$x(n) = x^*(n) \Rightarrow x(n)$ real $\forall n$
$\Rightarrow x$ is realvalued.

(ii) $y(k) = x(k/2)$ if k even; =0 otherwise.

$$\begin{aligned} y(\omega) &= \sum_{n=-\infty}^{\infty} y(n)e^{-i\omega n} \\ &= \sum_{neven} y(n)e^{-i\omega n} \\ &= \sum x(n/2)e^{-i\omega n} \\ &= \sum_{l=-\infty^{\infty}} x(l)e^{i2\omega l} \\ &= \sum_{l=-\infty^{\infty}} x(l)e^{i(2\omega)l} \\ &= x(2\omega) \end{aligned}$$

$\therefore y(\omega) = x(2\omega)$



Sketch of y: use the fact that $x(\omega)$ periodic with period 2π.

8

$$\begin{aligned}
\forall n \in \mathbb{N}, x(n) &= (0.5)^{|n|} \\
x(\omega) &= \sum_{m=-\infty}^{\infty} x(m) e^{-i\omega m} \\
&= \sum_{m=-\infty}^{\infty} (0.5)^{|m|} e^{-i\omega m} \\
&= \sum_{m=1}^{\infty} (0.5)^m (e^{-i\omega m} + e^{i\omega m}) + (0.5)^0 \\
&= \sum_{m=0}^{\infty} (0.5)^m ((e^{-i\omega})^m + (e^{i\omega})^m) - (0.5)^0 \\
&= \frac{1}{1-(0.5)(e^{-i\omega})} + \frac{1}{1-(0.5)(e^{i\omega})} - 1 \\
& \quad \text{[Using geometric series sum]} \\
&= \frac{e^{i\omega}}{e^{i\omega} - 0.5} + \frac{1}{1 - 0.5e^{i}\omega} - 1
\end{aligned}$$

9

$$\forall n \quad x(n) = \frac{1}{2\pi} \int_0^{2\pi} x(\omega) e^{i\omega n} d\omega$$

(a)

$$x(n) = e^{i\omega_0 n} = \frac{1}{2\pi} \int_0^{2\pi} x(\omega) e^{i\omega n} d\omega$$

$x(\omega)$ needs to pick out $e^{i\omega n}$ at $\omega = \omega_0$ with proper scaling.
$x(\omega) = 2\pi\delta(\omega - \omega_0) \quad \forall\omega \in [0, 2\pi]$
Note that DTFT $X(\omega)$ is always periodic with period 2π. Here we find expression for $x(\omega)$ only in $[0, 2\pi]$

(b)

$$y(n) = \cos(\omega_0, n) = \frac{1}{2} e^{i\omega_0 n} + \frac{1}{2} e^{i(-\omega_0)n}$$

$\omega_0 \in [0, 2\pi]$ but $-\omega_0 \notin [0, 2\pi]$, so use the fact that $e^{i(-\omega_0)n} = e^{i(-\omega n + 2\pi n)}$.

($e^{i\omega n}$ is periodic in ω with period 2π)

$$
\begin{aligned}
\therefore y(n) &= \frac{1}{2}e^{i\omega n} + \frac{1}{2}e^{i(2\pi-\omega_0)n} \\
&= \frac{1}{2\pi}\int_0^{2\pi} Y(\omega)e^{i\omega n}d\omega \\
Y(\omega) &= \pi\delta(\omega-(2\pi-\omega_0)) \quad \forall\omega\in[0,2\pi), \text{periodic period} 2\pi
\end{aligned}
$$

(c)

$$
\begin{aligned}
Z(n) &= e^{i\omega n}, \quad \omega_0 = 2\pi + \frac{\pi}{4} \\
Z(n) &= e^{i2\pi n}e^{i\frac{\pi}{4}n} \\
&= e^{i\frac{\pi}{4}n} \\
\therefore Z(\omega) &= 2\pi\delta(\omega-\frac{\pi}{4}) \quad \forall\omega\in[0,2\pi), \text{periodic period} 2\pi
\end{aligned}
$$

10 (a)

$$
x(t) = cos(20\pi t) = \frac{1}{2}e^{i20\pi t} + \frac{1}{2}e^{-i20\pi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)e^{i\omega t}d\omega
$$

$x(\omega)$ needs to pick out $e^{i\omega t}$ at $\omega = 20\pi$ and -20π, with some scaling $x(\omega) = \pi\delta(\omega-20\pi) + \pi\delta(\omega+20\pi)$

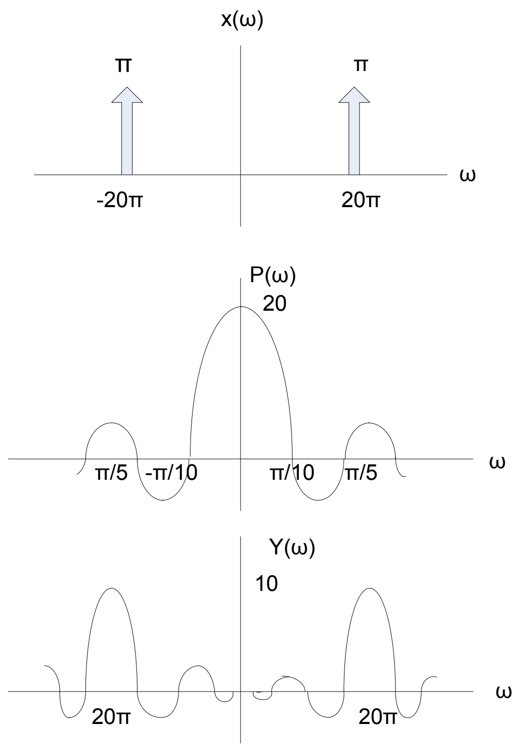
$$
\begin{aligned}
p(t) &= \begin{cases} 1 & -10 < t < 10 \\ 0 & \text{otherwise} \end{cases} \\
p(\omega) &= \int_{-\infty}^{\infty} p(t)e^{-i\omega t}dt \\
&= \int_{-10}^{10} e^{-i\omega t} \\
&= \frac{e^{-i\omega t}}{-i\omega}|_{-10}^{10} \\
&= \frac{1}{i\omega}(e^{i10\omega} - e^{i10\omega}) \\
p(\omega) &= \frac{2sin(10\omega)}{\omega} \\
y(t) &= x(t)p(t) \Rightarrow Y(\omega) = \frac{1}{2\pi}(x*p)(\omega)
\end{aligned}
$$

Note that $x(\omega)$ is sum of scaled, shifted impulses, so $(x*p)(\omega)$ would be just $p(\omega)$ shifted and scaled.

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi}(\pi p(\omega - 20\pi) + \pi p(\omega + 20\pi)) \\ &= \frac{sin(10(\omega - 20\pi))}{\omega - 20\pi} + \frac{sin(10(\omega + 20\pi))}{\omega + 20\pi} \end{aligned}$$

Unit of ω is rad/s

(b) Note that $x(\omega)$,$Y(\omega)$ are all real, so we can simply plot them (as opposed to plotting magnitude and phase in case of complex number)



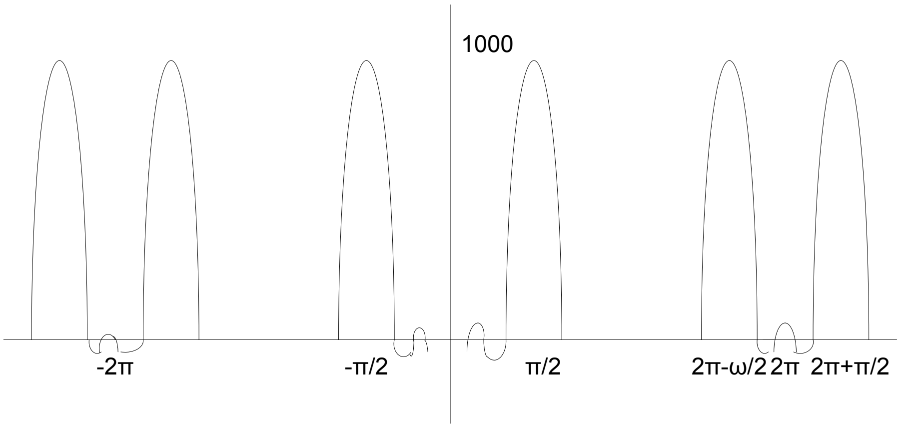
(π is the strength(area under) of the impulse)

$p(0)$ is actually undefined (divided by zero), but the limit is $\lim_{n\to\infty} \frac{2sin(10\omega)}{\omega}$ (by L'Hopital rule)=$\frac{2cos(10\omega)10}{1}|_{\omega=0} = 20$

$Y(0) = 0$ because $p(\omega) = 0$ when $\omega = \frac{\pi}{10}k, k \in \mathbb{Z}$ and 20π is divisible by $\frac{\pi}{10}$

(c) Unit of ω in discrete-time is rad/sample . Sampling has an effect of replicating $Y(\omega)$ every $2\pi f_s = 200\pi$rad/s and scale the frequency by f_s, so frequency $2\pi rad/sample$ in discrete time corresponds to frequency $2\pi f_s$ in continuous time. $z(\omega) = 100 \sum_{k=-\infty}^{\infty} Y(100(\omega - 2\pi k))$

(d) Note that there will be some tails from sines at other location although the effects are not significant. periodic with period 2π



11 (a)

$$t_1 = cos(10\pi t) \xrightarrow{F.T} x_1(\omega) = \pi\delta(\omega \pm 10\pi) \quad \omega_{B_1} = 10\pi$$

(b)

$$\begin{aligned} t_2 &= cos(20\pi t) \xrightarrow{F.T} x_2(\omega) = \pi\delta(\omega \pm 20\pi) \quad \omega_{B_2} = 20\pi \\ t_3 &= cos(30\pi t) \xrightarrow{F.T} x_3(\omega) = \pi\delta(\omega \pm 30\pi) \quad \omega_{B_3} = 30\pi \\ t_4 &= cos(10\pi t) + cos(20\pi t) + cos(30\pi t) \xrightarrow{F.T} \\ x_4(\omega) &= \pi(\delta(\omega \pm 10\pi) + \delta(\omega \pm 20\pi) + \delta(\omega \pm 30\pi)) \quad \omega_{B_4} = 30\pi \end{aligned}$$

(c)

$$\begin{aligned}
\omega_s &= 30\pi rad/s \leftrightarrow f_s = 15Hz \\
y_1(n) &= \cos(10\pi \frac{n}{15}) = \cos(\frac{2}{3}\pi n) \\
y_2(n) &= \cos(\frac{4}{3}\pi n) = \cos((-2\pi + \frac{4}{3}\pi)) \\
&= \cos(\frac{-2}{3}\pi n) = cos(\frac{2}{3}\pi n) \\
y_3(n) &= \cos(2\pi n) = 1 \\
y_4(n) &= 2\cos(\frac{2}{3}\pi n) + 1
\end{aligned}$$

(d)

$$\begin{aligned}
z_k(t)? \quad z_k(t)|_{t=\frac{n}{15}} &= y_k(n) \\
z_1(t) &= \cos(\frac{2}{3}\pi.15t) = cos(10\pi t) \quad \omega_B = 10\pi \\
z_2(t) &= \cos(10\pi t) \quad \omega_B = 10\pi \\
z_3(t) &= 1 \quad \omega_B = 0 \\
z_4(t) &= 2\cos(10\pi t) + 1 \quad \omega_B = 10\pi
\end{aligned}$$

Note: $z_(t) = cos((\omega + 2\pi k) f_s t)$ for any $k \in \mathbb{Z}$ will have the same $z(n) = cos(\omega n)$ after sampling at f_s. By making sure $|\omega + 2\pi k| < \pi$, we always have $z(t)$ with $\omega_B < \frac{f_s}{2}$.

12 (a) Note: If $x(n)$ is $z(t)$ sampled at sampling period T, $y(n)$ is just $z(t)$ sampled at sampling period $< T$.

Give the explanation on pg437 and final equation on pg440 on textbook,

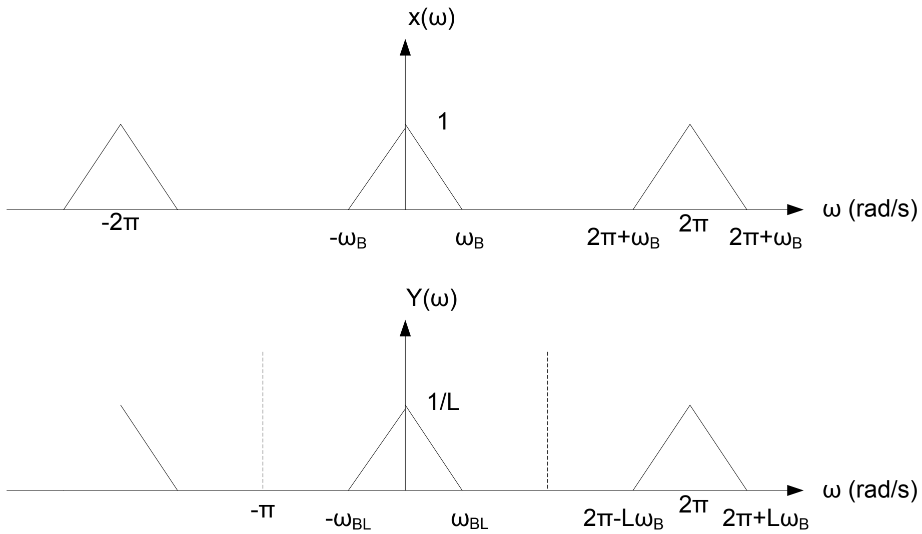
$$\begin{aligned}
X\omega) &= \frac{1}{T} \sum_{t=-\infty}^{\infty} Z(\frac{\omega}{T} - \frac{2\pi k}{T}) \\
Y(\omega) &= \frac{1}{LT} \sum_{t=-\infty}^{\infty} Z(\frac{\omega}{LT} - \frac{2\pi k}{LT}) \qquad (1)
\end{aligned}$$

where $X(\omega), Y(\omega)$ are $DTFT$ of $X(n)$ and $Y(n)$,and $Z(\omega)$ is $CTFT$ of

$Z(t)$, Now let $k = i + rL$ where $-\infty < r < \infty$ and $0 \le i \le L-1$

$$
\begin{aligned}
(1) \Longrightarrow Y(\omega) &= \frac{1}{LT}\sum_{i=0}^{L-1}\sum_{r=-\infty}^{\infty} Z(\frac{\omega}{LT} - \frac{2\pi(i+rL)}{LT}) \\
&= \frac{1}{L}\sum_{i=0}^{L-1}\frac{1}{T}\sum_{r=-\infty}^{\infty} Z(\frac{\omega - 2\pi i}{LT} - \frac{2\pi r}{T}) \\
&= \frac{1}{L}\sum_{i=0}^{L-1} X(\frac{\omega - 2\pi i}{L})
\end{aligned}
$$

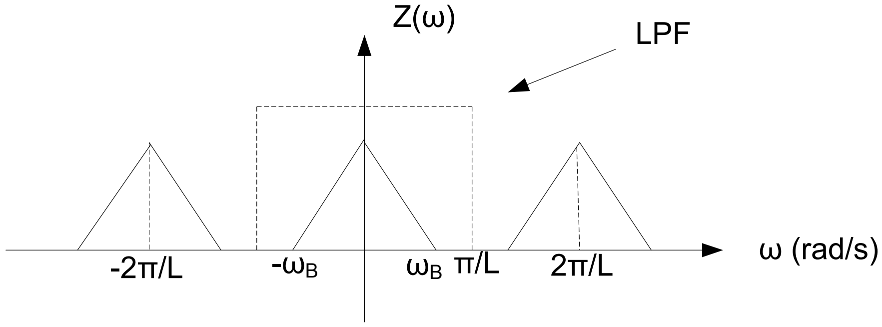
(b) We can reconstruct x only when there is no overlapping among $X(\frac{\omega - 2\pi i}{L})$



Therefore the condition is: $\omega_B L < \pi \Leftrightarrow \omega_B < \frac{\pi}{L}$ bandwidth of $x(n)$ is smaller than $\frac{\pi}{L}$

(c) Intuitively, we need to amplify $y(n)$ by L and increase the sampling rate.

$$\begin{aligned}
Let \quad Z(n) &= \begin{cases} Ly(n/L) & n = 0, \pm L, \pm 2L, ... \\ 0 & \text{otherwise} \end{cases} \\
Z(n) &= L \sum_{k=-\infty}^{\infty} y(k)\delta(n - kL) \\
Then \quad Z(\omega) &= \sum_{n=-\infty}^{\infty} (n)e^{-i\omega n} \\
&= L \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} y(k)\delta(n - kL)e^{-i\omega n} \\
&= L \sum_{k=-\infty}^{\infty} y(k)e^{-i\omega kL} \\
&= LY(\omega L)
\end{aligned}$$



clearly, after a low-pass filter with cut-off frequency at $\frac{\pi}{L}$, $\hat{Z}(\omega) = X(\omega)$,and $\hat{Z}(n) = X(n)$.

13 (a)

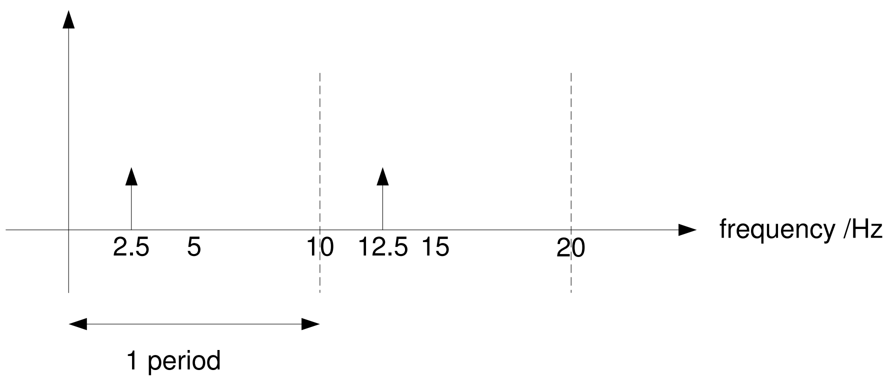
$$\begin{aligned}
W(n) &= Y(nT) \\
U(t) &= \sum_n W(n)\delta(t - nT) = \sum_n Y(nT)\delta(t - nT) \\
W(\omega) &= \frac{1}{T}\sum_k Y(\frac{\omega - 2\pi k}{T}) \\
U(\omega) &= W(\omega T) = \frac{1}{T}\sum_k Y(\omega - \frac{2\pi k}{T})
\end{aligned}$$

Note the convention that 'n' is for discrete time and 't' is for continuous time.

(b) We have $z(t)$ as the perfect reconstruction of $Y(t)$. $Z(\omega) = Y(\omega) = X(\omega)H(\omega)$

(c) $T = 0.1s \Rightarrow$ sampling frequency$= 10Hz. X(t) = \sin(25\pi t) + sin(5\pi t)$. $\therefore$ $Y(t) = Z(t) = \sin(5\pi t)$

(d) $H(\omega) = 1 \; \forall\omega \Longrightarrow Y(\omega)$ is not band-limited as before. Do we have aliasing? However,ata sampling frequency of $10Hz, 12.5Hz$ is indistinguishable from $2.5Hz$.

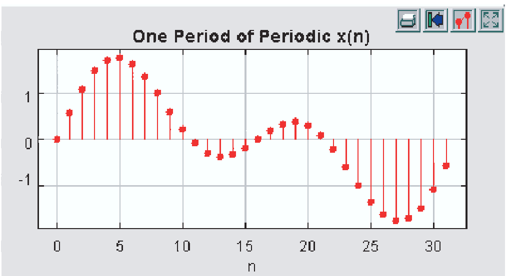


$\therefore Z(t) = \sin(5\pi t)$ as before.

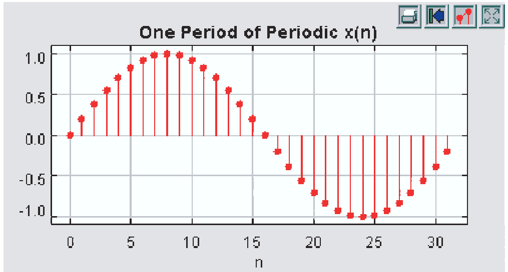
15 (a)

$$
\begin{aligned}
\text{show} \quad \cos(u+v) &= \cos u \cos v - \sin u \sin v \\
\cos u \cos v &= \frac{1}{4}(e^{ju} + e^{-ju})(e^{jv} + e^{-jv}) \\
&= \frac{1}{4}(e^{j(u+v)} + e^{-j(u+v)} + e^{-j(u-v)} + e^{j(u-v)}) \qquad (1) \\
\sin u \sin v &= -\frac{1}{4}(e^{ju} - e^{-ju})(e^{jv} - e^{-jv}) \\
&= -\frac{1}{4}(e^{j(u+v)} + e^{-j(u+v)} - e^{-j(u-v)} - e^{j(u-v)}) \qquad (2) \\
(1) - (2) &= \frac{1}{2}(e^{j(u+v)} + e^{-j(u+v)}) \\
&= \cos(u+v)
\end{aligned}
$$

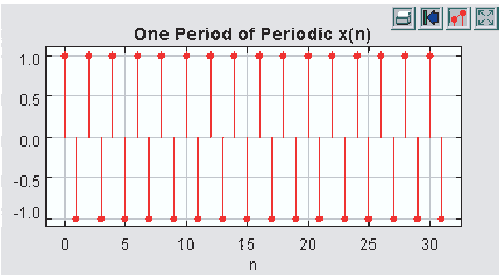
16 Suppose that x is a discrete-time signal with period $p = 32$. Below are plotted six possible such signals x. For each of these, match one of the six plots on the next page, or indicate that none matches.



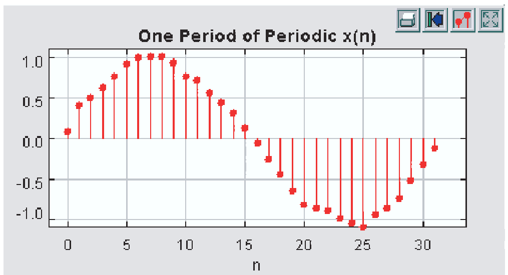
matching FS: d



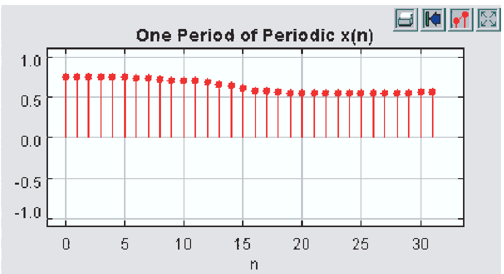
matching FS: a



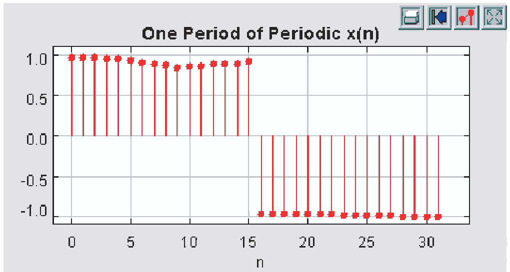
matching FS: n oe



matching FS: c



matching FS: e



matching FS: b

17 (a)

$$x(n) = 1 + \cos(\frac{\pi}{4}(n - p_1)) + \sin(\frac{\pi}{2}(n - p_2))$$

$$\left.\begin{array}{l} p_1 = 8 \\ p_2 = 4 \end{array}\right\} \Rightarrow p = 8sec$$

$$\omega_0 = \frac{2\pi}{p} = \frac{\pi}{4} rads/sec$$

(b)

$$\begin{aligned}
x(n) &= 1 + \cos(\frac{n\pi}{4}) + \cos(\frac{n\pi}{2} - \frac{\pi}{2}) \\
&= A_0 + \sum_{k=1}^{2} A_k \cos(k\omega_0 n + \phi_k) \\
k &= p/2 = 4; \\
A_0 &= 1, A_1 = 1, A_2 = 1, A_3 = A_4 = 0; \\
\phi_1 &= 0, \phi_2 = -\pi/2, \phi_3 = \phi_4 = 0
\end{aligned}$$

(c)

$$\begin{aligned}
x(n) &= \sum_{k=0}^{p-1} x_k e^{ik\omega_0 n} \qquad\qquad k = 4 \\
x_k &= \begin{cases} A_0 & \text{if} \quad k = 0 \\ A_k e^{i\phi_k/2} & \text{if} \quad k \in \{1, ...3\} \\ A_k e^{i\phi_k/2} + A_k e^{-i\phi_k/2} = A_k \cos(\phi) & \text{if} \quad k = 4 \\ A_{8-k} e^{-i\phi_{8-k}/2} & \text{if} \quad k \in \{5, ...7\} \end{cases} \\
x_0 &= 1 \\
x_1 &= A_1 e^{i\phi_1/2} = 1/2 \\
x_2 &= \frac{A_2 e^{i\phi_2}}{2} = \frac{e^{\frac{-\pi}{2} i}}{2} = \frac{\cos(\pi/2) - i\sin(\pi/2)}{2} = \frac{1}{2i} \\
x_3 &= 0 \\
x_4 &= 0 \\
x_5 &= 0 \\
x_6 &= -\frac{1}{2i} \\
x_7 &= \frac{1}{2}
\end{aligned}$$

(d)

$$\begin{aligned}
y(n) &= 1 + \sin(\frac{\pi n}{4}) - \cos(\frac{\pi n}{2}) \\
H'(\omega) &= 1 + iH(\omega), \quad H'(\omega) = \begin{cases} 2 & \text{if} \quad 0 < \omega < \pi \\ 1 & \text{if} \quad \omega = 0 \text{ or } \omega = \pi \\ 1 & \text{if} \quad -\pi < \omega < 0 \end{cases}
\end{aligned}$$

(f)

$$y(n) = 1 + e^{i\pi n/4} - ie^{i\pi n/2}$$

or

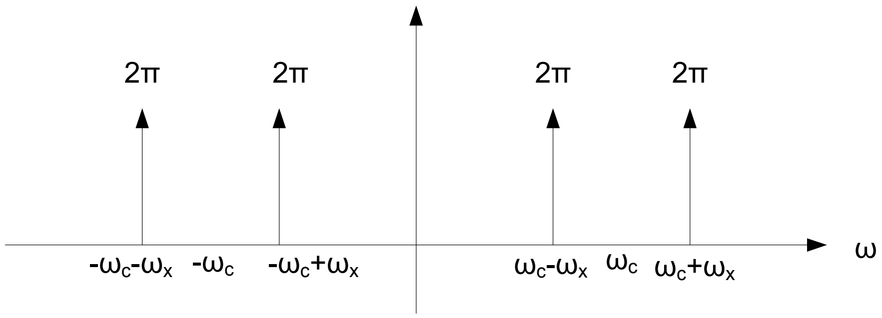
$$y(n) = 1 + e^{i\pi n/4} + ie^{i(\pi n/2 - \pi/2)}$$

18

$$\begin{aligned} \forall t \in, \quad c(t) &= 2\cos(\omega_c t) \\ \forall t \in, \quad x(t) &= 2\cos(\omega_x t) \end{aligned}$$

(a)

$$\begin{aligned} y(t) &= c(t)x(t) \\ Y(\omega) &= \frac{X(\omega - \omega_c) + X(\omega + \omega_c)}{2} \times 2 \\ X(\omega) &= \pi[\delta(\omega - \omega_x) + \delta(\omega + \omega_x)] \times 2 \\ Y(\omega) &= 2\pi[\delta(\omega - (\omega_c + \omega_x)) + \delta(\omega - (\omega_c - \omega_x)) + \delta(\omega + (\omega_c - \omega_x)) + \delta(\omega + (\omega_c + \omega_x) \end{aligned}$$



(b)

$$\begin{aligned} U(\omega) &= Y(\omega)H(\omega) \\ &= 2\pi[\delta(\omega - (\omega_c - \omega_x)) + \delta(\omega - (\omega_c + \omega_x))] \\ u(t) &= e^{j(\omega_c - \omega_x)t} + e^{j(\omega_c + \omega_x)t}, \quad \omega_c = 2\pi 8000, \omega_x = 2\pi 400 \end{aligned}$$

(c)

$$\begin{aligned} u'(n) &= e^{j\omega(2\pi 8000 - 2\pi 400).(\frac{n}{8000})} + e^{j\omega(2\pi 8000 + 2\pi 400).(\frac{n}{8000})} \\ &= e^{j\omega(2\pi n - 2\pi \frac{n}{20})} + e^{j\omega(2\pi n + 2\pi \frac{n}{20})} \\ &= e^{j\omega(-\frac{\pi n}{10})} + e^{j\omega(\frac{\pi n}{10})}; \qquad e^{j\omega(2\pi n)} = 1 \\ &= 2\cos(\frac{n\pi}{10}) \end{aligned}$$

(d)

$$z = \text{Ideal Interpolator}_T(u') \qquad\qquad T = \frac{1}{8000}s$$

$$\begin{aligned}
Z(\omega) &= U(\omega)S(\omega) \\
S(\omega) &= \begin{cases} T & \text{if} \quad |\omega| \leq \frac{\pi}{T} \\ 0 & \text{if} \quad |\omega| > \frac{\pi}{T} \end{cases} \\
W(t) &= \sum_{n=-\infty}^{\infty} u'(n)\delta(t - nT) \\
W(\omega) &= u'(\omega T) \qquad\qquad Z(\omega) = W(\omega)S(\omega) = u'(\omega T)S(\omega) \\
Z(\omega) &= \begin{cases} Tu'(\omega T) \quad \text{if} & \frac{-\pi}{T} < \omega \leq \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$