

Sampling and Reconstruction

Last time we

- Viewed aperiodic functions in terms of frequency components via Fourier transform
- Gained insight into the meaning of Fourier transform through comparison with Fourier series
- Defined (continuous-time) Fourier transform and DTFT and computed transforms for some aperiodic and periodic signals
- Identified useful properties of the Fourier transform

Today we will

- Explore sampling and reconstruction of signals
- See examples of the phenomenon known as aliasing
- Find the sampling rate needed for accurate reconstruction
- Learn about the Nyquist-Shannon Sampling theorem
- Describe different types of reconstruction

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Sampling

Let's define a system called Sampler that converts a continuous-time signal to a discrete-time signal by sampling every T seconds:

$$\text{Sampler}_T: [\text{Reals} \rightarrow \text{Complex}] \rightarrow [\text{Integers} \rightarrow \text{Complex}]$$

$$\forall x \in [\text{Reals} \rightarrow \text{Complex}] \text{ and } \forall n \in \text{Integers},$$

$$\text{Sampler}_T(x)(n) = x(nT)$$

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Sampling Sinusoids: Aliasing

Suppose we have a sinusoid with frequency f (in Hertz) that we want to sample.

$$x(t) = \cos(2\pi ft + \phi)$$

Sampling the signal at sampling frequency $f_s = 1/T$ yields

$$\text{Sampler}_T(x)(n) =$$

Notice that a sinusoid with frequency $f' = f + k f_s$ ($k \in \text{Integers}$) yields an **identical** sampled signal!

$$x'(t) = \cos(2\pi(f + k f_s)t + \phi)$$

$$\text{Sampler}_T(x')(n) =$$

Aliasing

- Thus, when we try to reconstruct a sinusoid from its samples, we can't know what frequency the original sinusoid had (unless we are given more information).
- Every discrete-time signal has many continuous-time "identities".
- We call this phenomenon **aliasing**.
- We want to reconstruct back to the original continuous signal, not an alias of the original signal.
- If we have more information about the original signal, e. g., we know that its frequency lies within some range, we may be able to rule out all but one of the aliases.

Nyquist Frequency

- Suppose we have samples of a sinusoid, and we want to reconstruct the original continuous-time sinusoid.
- If we know that the frequency f of the original continuous-time sinusoid lies within the interval $[0, \frac{1}{2} f_s)$, then we have ruled out all but one of the possibilities, and have uniquely identified the original signal.
- We can accurately reconstruct **any** continuous-time signal from samples as long as the sampling frequency f_s is at least **twice as large** as the **highest frequency component** in the original signal.
- The frequency $\frac{1}{2} f_s$ is known as the Nyquist frequency. It is the highest frequency we can reconstruct from a sampler.

Reconstruction

There are many different ways to reconstruct a signal.

- Zero-order hold: Create the continuous signal $x(t)$ by holding each sample $y(n)$ constant over the time period nT to $(n+1)T$.
- First-order hold: (Linear interpolation) For t between nT and $(n+1)T$, $x(t)$ varies linearly from $x(t)=y(n)$ at $t=nT$ to $x(t)=y(n+1)$ at $t=(n+1)T$.
- Ideal interpolation: The method that reconstructs sinusoids exactly. We will define this rigorously.

Each method of interpolation is itself a system, with domain [Integers \rightarrow Complex] and range [Reals \rightarrow Complex].

We will now mathematically define each of these systems.

Interpolation as Composition of Systems

We will represent each method of interpolation as the composition of a special system we will call ImpulseGen_T , and another system particular to the method of interpolation.

We define the system

$$\text{ImpulseGen}_T: [\text{Integers} \rightarrow \text{Complex}] \rightarrow [\text{Reals} \rightarrow \text{Complex}]$$

$$\forall y \in [\text{Integers} \rightarrow \text{Complex}] \text{ and } \forall t \in \text{Reals},$$

$$\text{ImpulseGen}_T(y)(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t-nT)$$

The system creates a signal with Dirac delta functions of height $y(n)$ at each time $t = nT$. Visualize it as a system that turns samples into Dirac delta functions.

Defining Methods of Interpolation

To perform the interpolation, the signal is first sent through ImpulseGen_T . The output of ImpulseGen_T is then sent through an LTI system that will determine the method of interpolation.

For **zero-order hold**, the LTI system has impulse response

$$h(t) = \begin{cases} 1 & \text{for } t \in [0, T) \\ 0 & \text{otherwise} \end{cases}$$

For **first-order hold (linear interpolation)**,

$$h(t) = \begin{cases} 1+t/T & t \in (-T, 0) \\ 1-t/T & t \in [0, T) \\ 0 & \text{otherwise} \end{cases}$$

For **ideal interpolation**, the impulse response is

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

Nyquist-Shannon Sampling Theorem

The ideal interpolation is important because it makes possible the perfect recovery of any signal as long as its highest frequency component is less than the Nyquist frequency.

The Nyquist-Shannon Sampling Theorem states:

If x is a continuous-time signal with Fourier transform X and if $X(\omega)$ is zero outside the range $-\pi/T < \omega < \pi/T$ rad/s, then

$$x = \text{IdealInterpolator}_T(\text{Sampler}_T(x))$$

where $\text{IdealInterpolator}_T$ is the composition of ImpulseGen_T and the LTI system defined on the previous slide.

Justification for the Sampling Theorem

While the formal proof for the theorem is rather technical (you can find it in a DSP book) we will give the intuition behind it:

Let's consider what Sampler_T and $\text{IdealInterpolator}_T$ do to a signal from a frequency standpoint.

The signal x passes through Sampler_T first, creating a signal y . Page 437 of the text shows that when a signal is sampled, the old CTFT X and new DTFT Y are related as follows:

$$Y(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

The CTFT of the original signal is copied and shifted by $2\pi k/T$.

If the original CTFT is zero outside the interval $-\pi/T < \omega < \pi/T$ then the copies won't overlap!

Justification for the Sampling Theorem

Now consider what ImpulseGen_T does to the sampled signal y .

$$w(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - nT)$$

According to page 437 in the text, the Fourier transforms of these two signals are related by

$$W(\omega) = Y(\omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi k}{T}\right)$$

Finally, consider what the LTI system does to the signal w . For

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} \quad \text{CTFT} \quad H(\omega) = \begin{cases} T & \text{for } \omega \in [-\pi/T, \pi/T] \\ 0 & \text{otherwise} \end{cases}$$

\Rightarrow

Justification for the Sampling Theorem

We know that if we pass a signal through an LTI system, the Fourier transform of the output is the product of the Fourier transform of the input and the frequency response.

So, the output z of the LTI system has Fourier transform

$$Z(\omega) = H(\omega)W(\omega) = \begin{cases} \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi k}{T}\right) & \text{for } \omega \in [-\pi/T, \pi/T] \\ 0 & \text{otherwise} \end{cases}$$

To make Z , we take copies of X , each shifted by $2\pi k/T$, and then cut away anything outside of the interval $-\pi/T < \omega < \pi/T$.

Z is the same as X if $X(\omega)$ was originally zero outside the interval $-\pi/T < \omega < \pi/T$.