

Stability and Z Transforms

Last time we

- Explored sampling and reconstruction of signals
- Saw examples of the phenomenon known as aliasing
- Found the sampling rate needed for accurate reconstruction
- Learned about the Nyquist-Shannon Sampling theorem
- Described different types of reconstruction

Today (and next time) we will

- Look at systems with impulse responses lacking a well-defined Fourier transform (unstable)
- Develop tools to analyze whether a system is stable: Z transform (for discrete-time systems) and Laplace transform (for continuous-time systems)
- Draw connections between these new transforms and Fourier transforms

Example: Unstable System

Recall our “bank account” system, defined by

$$y(n) = (1+r)y(n-1) + x(n)$$

where r is the interest rate (hopefully positive!)

The impulse response of this system is

$$h(n) = (1+r)^n$$

Recall that we said this system is **unstable**.

For a bounded input like the Kronecker delta function, we can get an output that grows without bound.

We want to be able to tell whether a system is bounded-input bounded-output **stable**. That is, we want to determine whether a bounded input can lead to an unbounded output.

Example: Unstable System

If we try to find the frequency response of the bank account system using the DTFT,

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n} = \sum_{n=-\infty}^{\infty} (1+r)^n e^{-i\omega n}$$

we see that the infinite sum will not converge if $r > 0$.

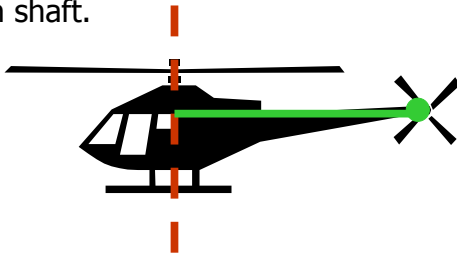
If we try to find the frequency response using the formula we developed for difference equations, we get

$$H(\omega) = \frac{1}{1 - (1+r)e^{-i\omega}}$$

This should make us wonder about the validity of this formula.

Another Example: Unstable System

Consider a helicopter. We can model the helicopter as a horizontal arm with moment of inertia M , which can rotate about the main shaft.



Let y be the rotational acceleration of the helicopter body, and let x be the torque applied to the helicopter.

(The torque actually comes from the torque created by the main rotor minus the counteracting torque from the tail rotor.)

Another Example: Unstable System

The motion is given by the differential equation

$$\frac{dy}{dt}(t) = x(t)/M$$

with solution (assuming zero initial condition)

$$y(t) = \frac{1}{M} \int_{\tau=0}^t x(\tau) d\tau$$

We can see that if $x(t)$ is some finite nonzero constant, the output $y(t)$ grows without bound as t increases.

Hence, the system is unstable.

Analyzing Stability: Discrete-Time

We now come up with ways of analyzing whether a system is stable. We want to know this, before we blindly use formulas and think we know the frequency response.

For a discrete-time LTI system, we say the system is **stable** if and only if its **impulse response is absolutely summable**.

Absolutely summable means that the sum

$$\sum_{n=-\infty}^{\infty} |h(n)|$$

exists and is finite.

This condition guarantees that the DTFT of h will exist.

Note that a system can be IIR and also be stable.

Analyzing Stability: Continuous-Time

For continuous-time LTI systems, the system is **stable** if and only if the **impulse response is absolutely integrable**.

Absolutely integrable means

$$\int_{t=-\infty}^{\infty} |h(t)| dt$$

exists and is finite.

In addition to this condition, the impulse response must have a finite number of local maxima/minima and a finite number of discontinuities on any finite interval, but real-life systems generally satisfy this so we don't talk about these conditions.

The Fourier transform of h exists if h is absolutely integrable.

Again, the system can be IIR and still be stable.

Determining Stability Using Z Transform

We can test whether a system is stable, and get some measure of how "close" it is to stability or instability, using the **Z transform**.

Consider a system whose impulse response is not absolutely summable. Maybe if we scale down the terms, say let

$$h_r(n) = h(n) r^{-n}$$

the new system h_r might be absolutely summable.

The **Z transform** of a signal does just this. It is defined by:

$$\hat{X}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Notice that it looks just like the DTFT, with $z = e^{i\omega}$.

Region of Convergence of Z Transform

For a given signal x , the Z transform

$$\hat{X}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

might not converge for all values of z . We define the **region of convergence (RoC)** as the set of complex numbers z where the above does converge:

$$\text{RoC}(x) = \{z \in \text{Complex} \mid x(n)|z|^{-n} \text{ is absolutely summable}\}$$

Z Transform of Impulse Response

The Z transform can tell us about the stability of a discrete-time LTI system, when we apply it to the impulse response.

The Z transform of the impulse response is known as the **transfer function** of the system, and it is a lot like the frequency response of the system.

$$\hat{H}(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

A discrete-time LTI system is **stable** if and only if the transfer function has a region of convergence that includes the unit circle.

The RoC gives us an idea of "how stable" the system is.

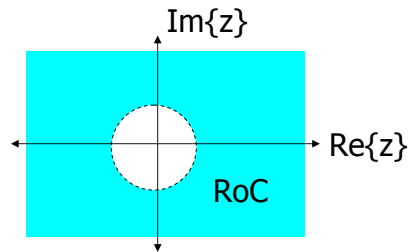
Causal Systems

For causal systems, $h(n) = 0$ for negative n . In this case, the transfer function sum only has terms for positive n :

$$\hat{H}(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

So if the transfer function RoC for a causal system contains a circle of radius r in the complex plane, it must also contain all circles of radius greater than r .

The RoC of the transfer function of a causal system always looks like this:



Poles and Zeroes

Difference equations have a transfer function with the form

$$\hat{H}(z) = \frac{A(z)}{B(z)} \quad \text{where } A \text{ and } B \text{ are each polynomials in } z.$$

The roots of the denominator of the transfer function are called **poles**. The roots of the numerator are called **zeroes**.

The places where the transfer function is infinite (the poles) determine the region of convergence.

As long as all the **poles lie strictly inside the unit circle**, the region of convergence will include the unit circle and the system will be **stable**. This is true only for **causal systems**.

