Frequency Response

Last time we

- Revisited formal definitions of linearity and time-invariance
- Found an eigenfunction for linear time-invariant systems
- Found the frequency response of a linear system to eigenfunction input
- Found the frequency response for cascade, feedback, difference equation, and differential equation systems

Today we will

- Extend the results to accommodate sinusoidal input, and then any input via Fourier series representation
- Write the Fourier series in terms of complex exponentials
- Provide a method to calculate Fourier series coefficients
- Determine properties of these coefficients

Response of an LTI System to Eigenfunction

Last time, we proved that for an input signal x given by

$$x(t) = e^{i\omega t} \quad \text{for all } t \in \text{Reals}$$

the corresponding output y of an LTI system can be expressed as

$$y(t) = H(\omega)\, e^{i\omega t} \quad \text{for all } t \in \text{Reals}$$

where $H(\omega)$ is called the **frequency response** of the system.

The complex exponential $e^{i\omega t}$ is called an **eigenfunction** of the system, because it creates an output with the same form, only differing by a scaling factor.

The same is true for a discrete input,

$$x(t) = e^{i\omega n} \quad \text{for all } n \in \text{Integers} \qquad \text{leads to}$$

$$y(n) = H(\omega)\, e^{i\omega n} \quad \text{for all } n \in \text{Integers}$$

Cosines as Complex Exponentials

Recall that cosines can be expressed as complex exponentials:

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

If we let $x_1(t) = e^{i\omega t}$ and $x_2(t) = e^{-i\omega t}$, we express the cosine as

$$\cos(\omega t) = ½(x_1(t) + x_2(t))$$

If we apply the cosine as input to an LTI system S, we find

$$S(½(x_1 + x_2)) = ½\ (S(x_1) + S(x_2))$$

and since x_1 and x_2 are eigenfunctions, we can write

$$\begin{aligned} y(t) &= ½\ (S(x_1) + S(x_2))(t) \\ &= ½\ (H(\omega)\ e^{i\omega t} + H(-\omega)\ e^{-i\omega t}\ ) \end{aligned}$$

So we can use frequency response to express the output for sinusoidal input.

Conjugate Symmetry

So for the input $x(t) = \cos(\omega t)$, we obtain the output

$$y(t) = ½\ (H(\omega)\ e^{i\omega t} + H(-\omega)\ e^{-i\omega t}\ ).$$

Realistic systems will produce purely real output (no imaginary component) for a purely real input like $\cos(\omega t)$.

This means that the imaginary parts of $H(\omega)\ e^{i\omega t}$ and $H(-\omega)\ e^{-i\omega t}$ must cancel out; they must be opposite in sign.

This is the same as saying that one is the conjugate of the other:

$$\begin{aligned} H(\omega)\ e^{i\omega t} &= (H(-\omega)\ e^{-i\omega t})^* \\ &= H(-\omega)^*\ e^{i\omega t} \end{aligned}$$

For systems that produce real output for real input, it is true that

$$H(\omega) = H(-\omega)^*$$

Implications: Scaled and Shifted Sinusoids

Systems that produce purely real output for a purely real input are called **conjugate symmetric**.

Let's look again at the output for our case $x(t) = \cos(\omega t)$,

$$y(t) = \tfrac{1}{2}\left(H(\omega)\, e^{i\omega t} + H(-\omega)\, e^{-i\omega t}\right)$$

Using the fact that $z + z^* = 2\,\mathrm{Re}\{z\}$,

$$\begin{aligned} y(t) &= \tfrac{1}{2}\left(2\,\mathrm{Re}\{H(\omega)\, e^{i\omega t}\}\right) \\ &= \mathrm{Re}\{H(\omega)\, e^{i\omega t}\} \end{aligned}$$

If we express $H(\omega)$ in polar form, $H(\omega) = |H(\omega)|\, e^{i\angle H(\omega)}$,

$$\begin{aligned} y(t) &= \mathrm{Re}\{|H(\omega)|\, e^{i\angle H(\omega)}\, e^{i\omega t}\} \\ &= \mathrm{Re}\{|H(\omega)|\, e^{i(\angle H(\omega)+\omega t)}\} \\ &= |H(\omega)| \cos(\omega t+\angle H(\omega)) \end{aligned}$$

Computing Sinusoidal Response

So, given the system response to an eigenfunction, $H(\omega)$, we can compute the **magnitude response** $|H(\omega)|$ and the **phase response** $\angle H(\omega)$.

These form the scaling factor and phase shift in the output, respectively.

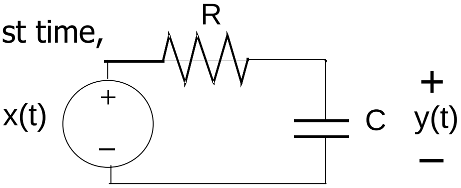
The frequency of the output sinusoid will be the same as the frequency of the input sinusoid in any LTI system.

The LTI system scales and shifts sinusoids.

These results hold true for both continuous and discrete signals and systems.

Example

Consider our RC circuit from last time, where we found



$$H(\omega) = \frac{1}{1 + i\omega RC}$$

To compute the voltage over the capacitor, y(t), for a sinusoidal input voltage x(t), I simply need to find the magnitude and phase of H(ω) and plug in:

$|H(\omega)| =$

$\angle H(\omega) =$

Response to Fourier Series Input

In Chapter 7, we mentioned that any periodic signal can be represented by a Fourier series:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$$

Since we are dealing with LTI systems, where we can pull out constants and distribute over sums, we can get the system output for any input by scaling and summing the output for the individual sinusoids in the Fourier series.

Alternate Fourier Series Representation

Remembering that

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

we may write

$$x(t) = A_0 + \sum_{k=1}^{\infty} \frac{A_k}{2}\left(e^{i(k\omega_0 t+\phi_k)} + e^{-i(k\omega_0 t+\phi_k)}\right)$$

and also

$$x(t) = A_0 + \sum_{k=1}^{\infty} \frac{A_k}{2} e^{i\phi_k} e^{ik\omega_0 t} + \frac{A_k}{2} e^{-i\phi_k} e^{-ik\omega_0 t}$$

and letting

$$X_k = \begin{cases} A_0 & \text{if } k = 0 \\ \tfrac{1}{2} A_k e^{i\phi_k} & \text{if } k > 0 \\ \tfrac{1}{2} A_{-k} e^{-i\phi_{-k}} & \text{if } k < 0 \end{cases}$$

we obtain

$$\boxed{x(t) = X_0 + \sum_{k=1}^{\infty} X_k e^{ik\omega_0 t} + X_{-k} e^{-ik\omega_0 t}}$$

Alternate Fourier Series Representation: Discrete

For a discrete periodic signal, with the new notation

$$X_k = \begin{cases} A_0 & \text{if } k = 0 \\ \tfrac{1}{2} A_k e^{i\phi_k} & \text{if } k < \lfloor p/2 \rfloor \\ A_k \cos(\phi_k) & \text{if } k = \lfloor p/2 \rfloor \\ \tfrac{1}{2} A_{p-k} e^{-i\phi_{p-k}} & \text{if } k > \lfloor p/2 \rfloor \end{cases}$$

$$\boxed{x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}}$$

The proof is given in the text on page 303.

Response to Fourier Series Input

Now let's apply a continuous input x(t) to an LTI system with frequency response H(ω) and find the output y(t):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

Due to linearity, we can distribute over the sum and pull out the constants X_k.

The result is a scaled sum of the output generated by each individual complex exponential $e^{ik\omega_0 t}$.

Since each $e^{i\omega t}$ has corresponding output $H(\omega)e^{i\omega t}$,

$$y(t) = \sum_{k=-\infty}^{\infty} X_k H(k\omega_0) e^{ik\omega_0 t}$$

Determining Fourier Series Coefficients

We now give formulae for the Fourier series coefficients for a periodic signal of period p:

For continuous signals x(t), t ∈ Reals:

$$X_m = \frac{1}{p}\int_0^p x(t)e^{-im\omega_0 t}\, dt$$

For discrete signals x(n) x(n), n ∈ Integers:

$$X_m = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-im\omega_0 n}$$

The textbook provides a validation of these formulae on page 306, but their derivation will be intuitive once we have covered Fourier transforms.