Review of Frequency Domain

Today we will review:

- Fourier series
  - why we use it
  - trig form & exponential form
  - how to get coefficients for each form
- Eigenfunctions
  - what they are
  - how they relate to LTI systems
  - how they relate to Fourier series
- Frequency response
  - what it represents
  - why we use it
  - how to find it
  - how to use it to find the output y for any input x
- Impulse response
  - what it represents
  - why we use it
  - how to find it
  - how to use it to find the output y for any input x

Fourier Series: Continuous

We can represent any **periodic** function x ∈ [Reals → Reals] using a sum called the Fourier series.

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$$ trigonometric form

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$ exponential form

We sometimes refer to the terms in the Fourier series as **frequency components**, since each term represents a sinusoid of frequency $k\omega_0$.

Fourier Series: Discrete

We can represent any **periodic** function $x \in$ [Integers → Reals] using a sum called the Fourier series.

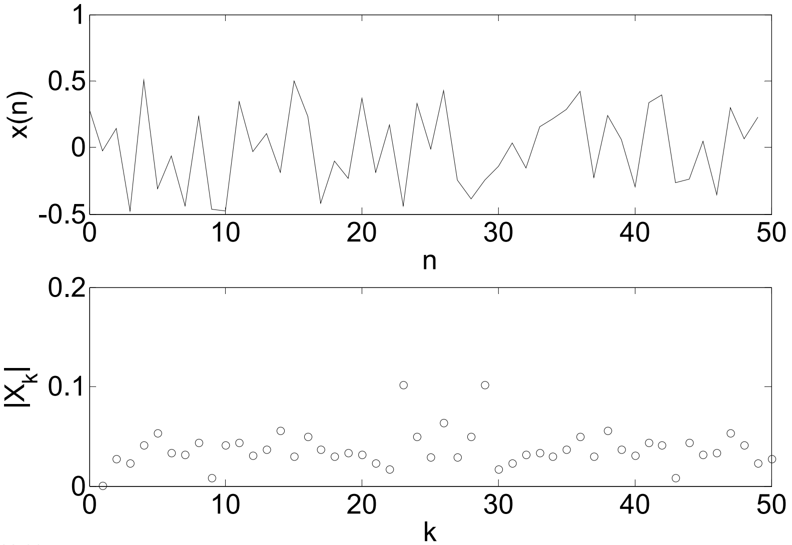
$$x(n) = A_0 + \sum_{k=1}^{\lfloor p/2 \rfloor} A_k \cos(k\omega_0 n + \phi_k) \qquad \text{trigonometric form}$$

$$x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n} \qquad \text{exponential form}$$

The sums in the Fourier series are finite for discrete-time signals, since discrete-time signals can only represent signals up to a certain maximum frequency which we will discuss in Chapter 11.

Fourier Series: Intuition

Signals with abrupt changes have high-frequency components.

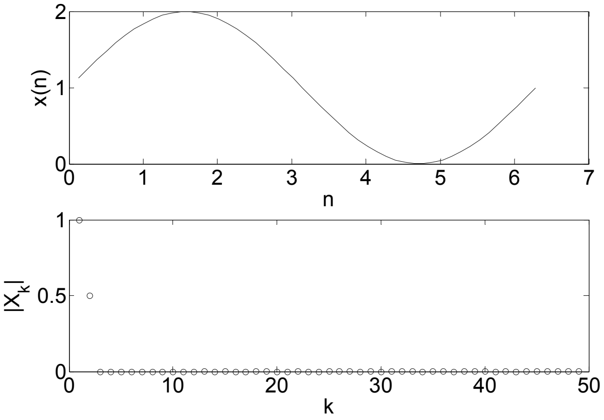


Fourier Series: Intuition

Smooth signals have small or zero high-frequency components.

Pure sinusoids centered at zero have only one nonzero Fourier coefficient: k=1 (the fundamental frequency) in the trig series.

The k=0 term represents the average value of the signal.



Fourier Series: Purpose

A Fourier series is another way to represent a signal, just like a graph, table, declarative definition, etc.

Useful things about the Fourier series representation:

- We can approximate a continuous signal using a finite number of terms from the series. This is another way to represent a continuous signal with finite data, like sampling.
- We can determine the **spectral** content of the signal: what frequencies it contains. This has practical applications: we may want to know if the signal is in the human audio range, etc.
- We can break the signal down in terms of eigenfunctions. This helps us see how an LTI system will transform the signal (what the output will be).

Finding the Fourier Series Coefficients

Two common ways to find the Fourier series coefficients:

1. Write x as a sum of cosines or complex exponentials and pick out the Fourier coefficients.

2. Use the integral formula to find the complex exponential Fourier series coefficients.

For continuous-time signals: $X_k = \frac{1}{p}\int_0^p x(t)e^{-ik\omega_0 t}\, dt$

For discrete-time signals: $X_k = \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ik\omega_0 n}$

Example

Find the trigonometric Fourier series for the following signal:

$$x(n) = \begin{cases} 1 & \text{for n odd} \\ 0 & \text{for n even} \end{cases}$$

Changing Between Fourier Series Forms

The trigonometric and complex exponential Fourier series forms are equivalent, and we can switch between forms.

Each X_k in the complex exponential Fourier series contains the amplitude A_k and phase shift φ_k in the form of a single complex number in polar form:

$$X_k = \begin{cases} A_0 & \text{if } k = 0 \\ \tfrac{1}{2} A_k e^{i\phi_k} & \text{if } k > 0 \\ \tfrac{1}{2} A_{-k} e^{-i\phi_{-k}} & \text{if } k < 0 \end{cases}$$

Continuous Signals

$$X_k = \begin{cases} A_0 & \text{if } k = 0 \\ \tfrac{1}{2} A_k e^{i\phi_k} & \text{if } k < \left\lfloor p/2 \right\rfloor \\ A_k \cos(\phi_k) & \text{if } k = \left\lfloor p/2 \right\rfloor \\ \tfrac{1}{2} A_{p-k} e^{-i\phi_{p-k}} & \text{if } k > \left\lfloor p/2 \right\rfloor \end{cases}$$

Discrete Signals

Example

Suppose the complex exponential Fourier series for a certain signal is given by

$X_0 = 0$ $X_k = (3+4i)/k$ for $k > 0$ $X_k = (4i-3)/k$ for $k < 0$

Find the trigonometric Fourier series.

Eigenfunctions

- One of the reasons the Fourier series is so important is that it represents a signal in terms of eigenfunctions of LTI systems.
- When I put a complex exponential function like $x(t) = e^{i\omega t}$ through a linear time-invariant system, the output is

  $$y(t) = S(x)(t) = H(\omega)\, e^{i\omega t}$$

  where $H(\omega)$ is a complex constant (it does not depend on time).
- The LTI system scales the complex exponential $e^{i\omega t}$.
- We call the complex exponential an eigenfunction. The LTI system S scales the function but does not change its form.
- Each system has its own constant $H(\omega)$ that describes how it scales eigenfunctions. It is called the **frequency response**.
- The frequency response $H(\omega)$ does not depend on the input. It is another way to describe a system, like (A, B, C, D), h, etc.
- If we know $H(\omega)$, it is easy to find the output when the input is an eigenfunction. $y(t)=H(\omega)x(t)$ true **when x is eigenfunction**!

Finding the Output for any Input via Fourier Series

Finding the output for an input signal is easy for complex exponential (eigenfunction) input. However, most input signals that we deal with are not complex exponentials.

We can take advantage of this easy input/output relationship that complex exponentials have by writing any old input signal x in terms of complex exponentials via Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \qquad\qquad x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n}$$

Then, the output will be the sum of all the responses to all the individual complex exponential terms, each with frequency $k\omega_0$:

$$y(t) = \sum_{k=-\infty}^{\infty} X_k H(k\omega_0) e^{ik\omega_0 t} \qquad\qquad y(n) = \sum_{k=0}^{p-1} X_k H(k\omega_0) e^{ik\omega_0 n}$$

Example

Consider a system with transfer function $H(\omega)=e^{i\omega\pi/8}$.

Find the system output for the input signal with period π and Fourier coefficients

$X_1 = 3i$ $X_{-1} = -3i$ $X_k = 0$ for all other k in Integers

Finding the Frequency Response

We can begin to take advantage of this way of finding the output for any input once we have $H(\omega)$.

To find the frequency response $H(\omega)$ for a system, we can:

1. Put the input $x(t) = e^{i\omega t}$ into the system definition
2. Put in the corresponding output $y(t) = H(\omega)\, e^{i\omega t}$
3. Solve for the frequency response $H(\omega)$.

   (The terms depending on t will cancel.)

We also have some other tools, like cascading systems, Mason's rule for feedback systems, formulae for difference and differential equations, etc.

Example

Find H(ω) for the system whose input-output relationship is defined for all t in Reals by

$$\frac{dy}{dt}(t) = 3y(t) + 2x(t)$$

Cosine Input to LTI System

A cosine is a common input function that we will consider.

It can be written as a sum of complex exponentials:

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

As a result, the cosine is "almost" an eigenfunction of an LTI system. A cosine is **scaled** and **phase shifted** by an LTI system:

For x given by $x(t) = \cos(\omega t)$ for all $t \in$ Reals,

$$y(t) = S(x)(t) = |H(\omega)| \cos(\omega t + \angle H(\omega))$$

The scaling factor is the magnitude of the frequency response, and the phase shift is the angle of the frequency response.

The same holds true for discrete-time systems as well.

Example

Consider the system with frequency response $H(\omega) = \frac{2}{i\omega - 3}$

Find the output y for the input given by x(t) = cos(4t).

Importance of Frequency Response and Impulse Response

- The frequency response is a way to define a system in terms of its reaction to periodic inputs of certain frequencies. This has many practical applications such as filter design.
- The frequency response can be used to quickly find the output for a given input when the input is a complex exponential, sinusoidal, or expressed via Fourier series.
- When we design a system to meet a frequency response specification, we need some way to have the system perform its action on a time-domain signal. We can express this action using time convolution with the **impulse response**.
- The frequency response and impulse response are both ways to define a system.

The Impulse Response

For discrete-time systems, the impulse response h is the particular system output obtained when the input is the Kronecker delta function

$$\forall\, n \in \text{Integers}, \qquad \delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

For continuous systems, the impulse response h is the particular output obtained when the input is the Dirac delta function δ, defined to have following properties:

$$\forall\, t \in \text{Reals} \setminus \{0\}, \qquad \delta(t) = 0$$

$$\forall\, \varepsilon \in \text{Reals with } \varepsilon > 0, \qquad \int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1$$

From Impulse Response to Frequency Response

For continuous-time systems, the frequency response is the **Fourier transform** of the impulse response:

$$H(\omega) = \int_{\tau=-\infty}^{\infty} h(\tau) e^{-i\omega\tau}\, d\tau$$

For discrete-time systems, the frequency response is the **discrete-time Fourier transform** (DTFT) of the impulse response:

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-i\omega k}$$

Example

Consider the continuous-time system which takes the average value of an input over 5 time units:

$$y(t) = \frac{1}{5}\int_{\tau=0}^{5} x(t-\tau)d\tau$$

Find the impulse response, and find the frequency response via Fourier transform.

Finding the Output for Any Input Using Convolution with Impulse Response

The impulse response gives us the output for any input via **convolution** (and in this way **defines the system**):

For $x \in$ [Reals → Reals],
$\forall t \in$ Reals,

$$y(t) = (h * x)(t) \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

For $x \in$ [Integers → Integers],
$\forall n \in$ Integers,

$$y(n) = (h * x)(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Recall that the roles of h and x in the above may be reversed.

If the system is **causal**, that is, if the output y(n) does not depend on future values of the input x(n+m) for m > 0, then the impulse response h(n) is zero for n < 0.

Example

Consider a system with impulse response

$$h(t) = \begin{cases} \frac{1}{5} & \text{for } t \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

Find the output corresponding to the input x(t) = cos(10 t).

Some Things You Need To Know

- Multiple ways of finding how to get output y for an input x
  - Special cases of sinusoidal and eigenfunction input
  - Convolution with h, using H(ω), using (A, B, C, D)
- Find and interpret the Fourier series for a signal
  - Using the integral method and simpler "eyeball" method
  - Reality check results using smoothness, even/odd-ness
- Find various system descriptors for LTI systems
  - Find (A, B, C, D) for a system
  - Find H(ω) using eigenfunctions as input, or Fourier transform of h, or previously derived properties/equations
  - Find h using system definition with Delta functions as input, or using (A, B, C, D)
- Demonstrate understanding of linearity, time-invariance, causality, determine whether systems have these properties