Fundamentals of Computer Systems

Thinking Digitally

Stephen A. Edwards

Columbia University

Fall 2011

The Subject of this Class

0

The Subjects of this Class

0 1

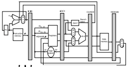
Engineering Works Because of Abstraction

Application Software

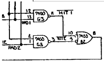
Operating Systems

```
  ;; voice 1 wave select
ld      a,(#CH1_W_NUM)
and     a
ld      a,(#CH1_W_SEL)
jr      nz,#00b4
ld      a,(#CH1_E_TABLE0)
```

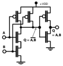
Architecture



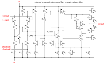
Micro-Architecture



Logic

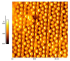


Digital Circuits



Analog Circuits

Devices

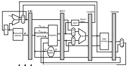
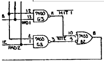
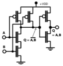
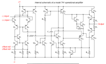
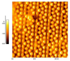


Physics

Engineering Works Because of Abstraction

	Application Software	COMS 3157, 4156, et al.
	Operating Systems	COMS W4118

`;; voice 1 wave select`
`ld a,(#CH1_W_NUM)`
`and a`
`ld a,(#CH1_W_SEL)`
`jr nz,#00b4`
`ld a,(#CH1_E_TABLE0)`
	Architecture	Second Half of 3827

	Micro-Architecture	Second Half of 3827
	Logic	First Half of 3827
	Digital Circuits	First Half of 3827
	Analog Circuits	ELEN 3331
	Devices	ELEN 3106
	Physics	ELEN 3106 et al.

Boring Stuff

`http://www.cs.columbia.edu/~sedwards/classes/2011/3827-fall/`

Prof. Stephen A. Edwards

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Class meets 10:35–11:50 AM Tuesdays and Thursdays in 633 Mudd

Lectures Sep 6–Dec 8

Holidays: Nov 8, Nov 24

Assignments and Grading

Weight	What	When
40%	Six homeworks	See Webpage
30%	Midterm exam	October 25th
30%	Final exam	9–12, December 20th

Homework is due at the beginning of lecture.

I will drop the lowest of your six homework scores; you

$$\text{can} \left\{ \begin{array}{c} \text{skip} \\ \text{omit} \\ \text{forget} \\ \text{ignore} \\ \text{blow off} \\ \text{screw up} \\ \text{feed to dog} \\ \text{flake out on} \\ \text{sleep through} \end{array} \right\} \text{one with no penalty.}$$

Rules and Regulations

You may collaborate with classmates on homework.

Each paper turned in must be unique; work must ultimately be your own.

List your collaborators on your homework.

Don’t cheat: if you’re stupid enough to try, we’re smart enough to catch you.

Tests will be closed-book with a one-page “cheat sheet” of your own devising.

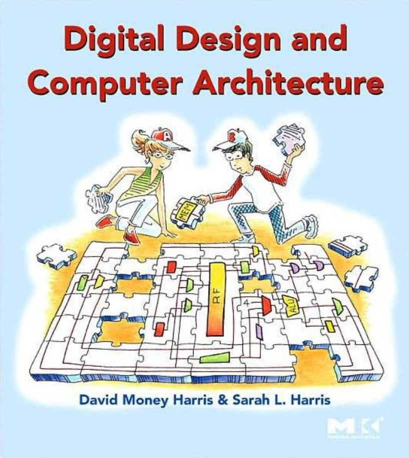
The Text

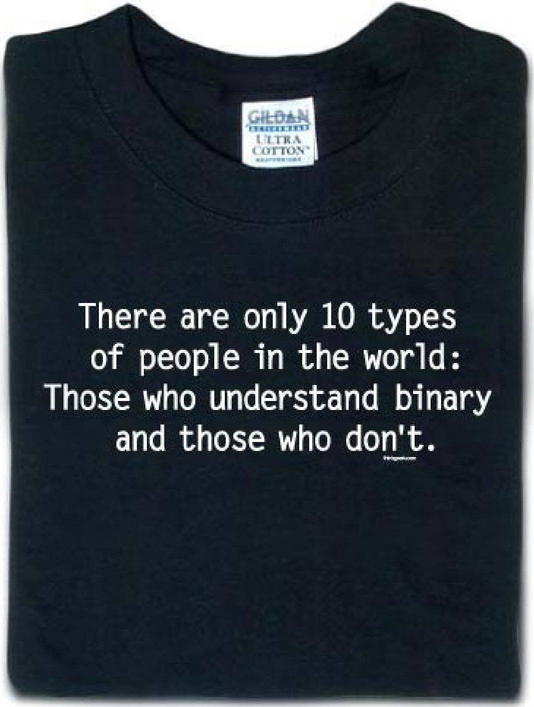
David Harris and Sarah Harris.

Digital Design and Computer Architecture.

Morgan-Kaufmann, 2007.

Almost precisely right for the scope of this class: digital logic and computer architecture





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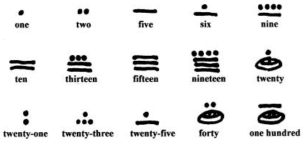
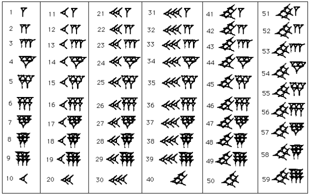
Which Numbering System Should We Use?
Some Older Choices:

Roman: I II III IV V VI VII VIII IX X



Mayan: base 20, Shell = 0



Babylonian: base 60

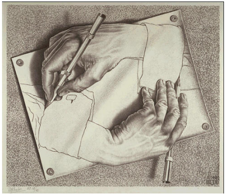
The Decimal Positional Numbering System

Ten figures: 0 1 2 3 4 5 6 7 8 9

$$7 \times 10^2 + 3 \times 10^1 + 0 \times 10^0 = 730_{10}$$

$$9 \times 10^2 + 9 \times 10^1 + 0 \times 10^0 = 990_{10}$$

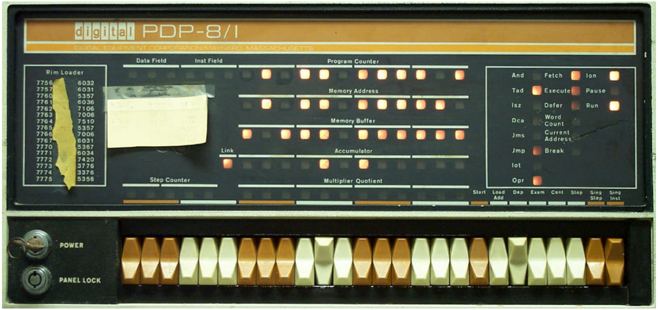
Why base ten?



Hexadecimal, Decimal, Octal, and Binary

Hex	Dec	Oct	Bin
0	0	0	0
1	1	1	1
2	2	2	10
3	3	3	11
4	4	4	100
5	5	5	101
6	6	6	110
7	7	7	111
8	8	10	1000
9	9	11	1001
A	10	12	1010
B	11	13	1011
C	12	14	1100
D	13	15	1101
E	14	16	1110
F	15	17	1111

Binary and Octal



DEC PDP-8/I, c. 1968

Oct	Bin
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111

$$
\begin{aligned}
\text{PC} &= 0\times 2^{11} + 1\times 2^{10} + 0\times 2^{9} + 1\times 2^{8} + 1\times 2^{7} + 0\times 2^{6} + \\
&\quad 1\times 2^{5} + 1\times 2^{4} + 1\times 2^{3} + 1\times 2^{2} + 0\times 2^{1} + 1\times 2^{0} \\
&= 2\times 8^{3} + 6\times 8^{2} + 7\times 8^{1} + 5\times 8^{0} \\
&= 1469_{10}
\end{aligned}
$$

Hexadecimal Numbers

Base 16: 0 1 2 3 4 5 6 7 8 9 A B C D E F

Instead of groups of 3 bits (octal), Hex uses groups of 4.

$$
\begin{aligned}
\mathrm{CAFEF00D}_{16} &= 12 \times 16^7 + 10 \times 16^6 + 15 \times 16^5 + 14 \times 16^4 + \\
&\quad 15 \times 16^3 + 0 \times 16^2 + 0 \times 16^1 + 13 \times 16^0 \\
&= 3,405,705,229_{10}
\end{aligned}
$$

| C | A | F | E | F | 0 | 0 | D | Hex

11001010111111101111000000001101 Binary

| 3 | 1 | 2 | 7 | 7 | 5 | 7 | 0 | 0 | 1 | 5 | Octal

Computers Rarely Manipulate True Numbers

Infinite memory still very expensive

Finite-precision numbers typical

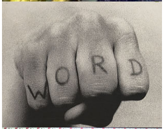
32-bit processor: naturally manipulates 32-bit numbers

64-bit processor: naturally manipulates 64-bit numbers

How many different numbers can you represent with 5 digits?

- binary
- octal
- decimal
- hexadecimal

Jargon

	Bit	Binary digit: 0 or 1
	Byte	Eight bits
	Word	Natural number of bits for the processor, e.g., 16, 32, 64
	LSB	Least Significant Bit (“rightmost”)
	MSB	Most Significant Bit (“leftmost”)

Decimal Addition Algorithm

$$\begin{array}{r} 434 \\ +628 \\ \hline \end{array}$$

$$4 + 8 \quad = \quad 12$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Decimal Addition Algorithm

$$\begin{array}{r} 1 \\ 434 \\ +628 \\ \hline 2 \end{array}$$

$$4 + 8 = 12$$

$$1 + 3 + 2 = 6$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Decimal Addition Algorithm

$$\begin{array}{r} {\color{red}1} \\ 434 \\ +628 \\ \hline {\color{blue}62} \end{array}$$

$$\begin{aligned} 4+8 &= {\color{red}1}{\color{blue}2} \\ 1+3+2 &= {\color{blue}6} \\ 4+6 &= {\color{red}1}{\color{blue}0} \end{aligned}$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Decimal Addition Algorithm

$$
\begin{array}{r}
\color{red}{1\;\,1\;\;\,} \\
434 \\
+628 \\
\hline
\color{blue}{062}
\end{array}
$$

$$
\begin{aligned}
4 + 8 &= \color{red}{1}\color{blue}{2} \\
1 + 3 + 2 &= \color{blue}{6} \\
4 + 6 &= \color{red}{1}\color{blue}{0}
\end{aligned}
$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Decimal Addition Algorithm

$$\begin{array}{r} \color{red}{1\ \ 1\ \ } \\ 434 \\ +628 \\ \hline \color{blue}{1062} \end{array}$$

$$\begin{aligned} 4+8 &= \color{red}{1}\color{blue}{2} \\ 1+3+2 &= \color{blue}{6} \\ 4+6 &= \color{red}{1}\color{blue}{0} \end{aligned}$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Binary Addition Algorithm

$$\begin{array}{r} 10011 \\ +11001 \\ \hline \end{array}$$

$$1 + 1 \quad = \quad {\color{red}1}{\color{blue}0}$$

+	0	1
0	00	01
1	01	10
10	10	11

Binary Addition Algorithm

$$\begin{array}{r} \color{red}{1} \\ 10011 \\ +11001 \\ \hline \color{blue}{0} \end{array}$$

$$1 + 1 \quad = \quad \color{red}{1}\color{blue}{0}$$

$$1 + 1 + 0 \quad = \quad \color{red}{1}\color{blue}{0}$$

+	0	1
0	00	01
1	01	10
10	10	11

Binary Addition Algorithm

$$\begin{array}{r} \color{red}{11} \\ 10011 \\ +11001 \\ \hline \color{blue}{00} \end{array}$$

$$1 + 1 \quad = \quad \color{red}{1}\color{blue}{0}$$

$$1 + 1 + 0 \quad = \quad \color{red}{1}\color{blue}{0}$$

$$1 + 0 + 0 \quad = \quad \color{red}{0}\color{blue}{1}$$

+	0	1
0	00	01
1	01	10
10	10	11

Binary Addition Algorithm

$$\begin{array}{r} \color{red}{011} \\ 10011 \\ +11001 \\ \hline \color{blue}{100} \end{array}$$

$$\begin{aligned} 1 + 1 &= \color{red}{1}\color{blue}{0} \\ 1 + 1 + 0 &= \color{red}{1}\color{blue}{0} \\ 1 + 0 + 0 &= \color{red}{0}\color{blue}{1} \\ 0 + 0 + 1 &= \color{red}{0}\color{blue}{1} \end{aligned}$$

+	0	1
0	00	01
1	01	10
10	10	11

Binary Addition Algorithm

$$\begin{array}{r} 0011 \\ 10011 \\ +11001 \\ \hline 1100 \end{array}$$

$$\begin{aligned} 1 + 1 &= 10 \\ 1 + 1 + 0 &= 10 \\ 1 + 0 + 0 &= 01 \\ 0 + 0 + 1 &= 01 \\ 0 + 1 + 1 &= 10 \end{aligned}$$

+	0	1
0	00	01
1	01	10
10	10	11

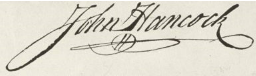
Binary Addition Algorithm

$$
\begin{array}{r}
10011 \\
10011 \\
+11001 \\
\hline
101100
\end{array}
$$

$$
\begin{aligned}
1 + 1 &= 10 \\
1 + 1 + 0 &= 10 \\
1 + 0 + 0 &= 01 \\
0 + 0 + 1 &= 01 \\
0 + 1 + 1 &= 10
\end{aligned}
$$

+	0	1
0	00	01
1	01	10
10	10	11

Signed Numbers: Dealing with Negativity



How should both positive and negative numbers be represented?

Signed Magnitude Numbers

You are most familiar with this: negative numbers have a leading $-$

In binary, a leading 1 means negative:

$0000_2 = 0$

$0010_2 = 2$

$1010_2 = -2$

$1111_2 = -7$

$1000_2 = -0?$

Can be made to work, but addition is annoying:

If the signs match, add the magnitudes and use the same sign.

If the signs differ, subtract the smaller number from the larger; return the sign of the larger.

One’s Complement Numbers

Like Signed Magnitude, a leading 1 indicates a negative One’s Complement number.

To negate a number, complement (flip) each bit.

$0000_2 = 0$

$0010_2 = 2$

$1101_2 = -2$

$1000_2 = -7$

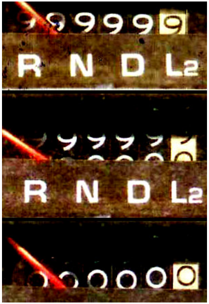
$1111_2 = -0?$

Addition is nicer: just add the one’s complement numbers as if they were normal binary.

Really annoying having a -0: two numbers are equal if their bits are the same or if one is 0 and the other is -0.


NOT ALL
ZEROS
ARE CREATED
EQUAL
0 CALORIE COLA
pepsi MAX
zero calories
maximum taste
ZERO CALORIES. MAXIMUM PEPSI® TASTE.
pepsi MAX
zero calories

Two’s Complement Numbers



Really neat trick: make the most significant bit represent a *negative* number instead of positive:

$1101_2 = -8 + 4 + 1 = -3$

$1111_2 = -8 + 4 + 2 + 1 = -1$

$0111_2 = 4 + 2 + 1 = 7$

$1000_2 = -8$

Easy addition: just add in binary and discard any carry.

Negation: complement each bit (as in one’s complement) then add 1.

Very good property: no -0

Two’s complement numbers are equal if all their bits are the same.

Number Representations Compared

Bits	Binary	Signed Mag.	One's Comp.	Two's Comp.
0000	0	0	0	0
0001	1	1	1	1
⋮				
0111	7	7	7	7
1000	8	−0	−7	−8
1001	9	−1	−6	−7
⋮				
1110	14	−6	−1	−2
1111	15	−7	−0	−1

Smallest number

Largest number

Fixed-point Numbers

How to represent fractional numbers? In decimal, we continue with negative powers of 10:

$$\begin{aligned} 31.4159 &= 3 \times 10^1 + 1 \times 10^0 + \\ &\quad 4 \times 10^{-1} + 1 \times 10^{-2} + 5 \times 10^{-3} + 9 \times 10^{-4} \end{aligned}$$

The same trick works in binary:

$$\begin{aligned} 1011.0110_2 &= 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 + \\ &\quad 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4} \\ &= 8 + 2 + 1 + 0.25 + 0.125 \\ &= 11.375 \end{aligned}$$

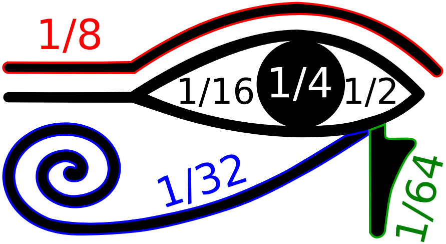
F

F a

u c

Interesting

The ancient Egyptians used binary fractions:



The Eye of Horus

Floating-point Numbers

How can we represent very large and small numbers with few bits?

Floating-point numbers: a kind of scientific notation

IEEE-754 floating-point numbers:

$$\underbrace{1}_{\text{sign}}\ \underbrace{10000001}_{\text{exponent}}\ \underbrace{01100000000000000000000}_{\text{significand}}$$

$$= -1.011_2 \times 2^{129-127}$$

$$= -1.375 \times 4$$

$$= -5.5$$

Binary-Coded Decimal

thinkgeek.com

Humans prefer reading decimal numbers; computers prefer binary.

BCD is a compromise: every four bits represents a decimal digit.

Dec	BCD
0	0000 0000
1	0000 0001
2	0000 0010
⋮	⋮
8	0000 1000
9	0000 1001
10	0001 0000
11	0001 0001
⋮	⋮
19	0001 1000
20	0010 0000
⋮	⋮

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

$$
\begin{array}{r}
158 \\
+242 \\
\hline
\end{array}
$$

$$
\begin{array}{rrrl}
0001 & 0101 & 1000 & \\
+0010 & 0100 & 0010 & \\
\hline
 & & 1010 & \text{First group}
\end{array}
$$

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

$$\begin{array}{r} 158 \\ +242 \\ \hline \end{array}$$

$$\begin{array}{rrrl} 0001 & 0101 & 1000 & \\ +0010 & 0100 & 0010 & \\ \hline & & 1010 & \text{First group} \\ & & +\ 0110 & \text{Correction} \\ \hline \end{array}$$

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

$$
\begin{array}{r}
\color{red}{1} \\
158 \\
+242 \\
\hline
0
\end{array}
$$

$$
\begin{array}{rrrl}
 & \color{red}{1} & & \\
0001 & 0101 & 1000 & \\
+0010 & 0100 & 0010 & \\
\hline
 & & 1010 & \text{First group} \\
 & & +\ 0110 & \text{Correction} \\
\cline{2-3}
 & 1010 & 0000 & \text{Second group}
\end{array}
$$

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

$$
\begin{array}{r}
1 \\
158 \\
+242 \\
\hline
0
\end{array}
$$

$$
\begin{array}{rrrl}
 & 1 & & \\
0001 & 0101 & 1000 & \\
+0010 & 0100 & 0010 & \\
\hline
 & & 1010 & \text{First group} \\
 & + & 0110 & \text{Correction} \\
\cline{2-3}
 & 1010 & 0000 & \text{Second group} \\
+ & 0110 & & \text{Correction} \\
\cline{1-2}
 & & & \\
\hline
\end{array}
$$

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

$$
\begin{array}{r}
{\color{red}11} \\
158 \\
+242 \\
\hline
00
\end{array}
$$

$$
\begin{array}{rrrrl}
 & {\color{red}1} & {\color{red}1} & & \\
 & 0001 & 0101 & 1000 & \\
+ & 0010 & 0100 & 0010 & \\
\hline
 & & & 1010 & \text{First group} \\
 & & + & 0110 & \text{Correction} \\
\cline{3-4}
 & & 1010 & 0000 & \text{Second group} \\
 & + & 0110 & & \text{Correction} \\
\cline{1-3}
 & 0100 & 0000 & & \text{Third group} \\
 & & & & \\
\hline
\end{array}
$$

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

$$\begin{array}{r} \color{red}{11} \\ 158 \\ +242 \\ \hline 400 \end{array}$$

	$\color{red}{1}$	$\color{red}{1}$		
	0001	0101	1000	
+	0010	0100	0010	
			1010	First group
		+	0110	Correction
		1010	0000	Second group
	+	0110		Correction
	0100	0000		Third group
				(No correction)
	0100	0000	0000	Result