Functors and Friends

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Functors: Types That Hold a Type in a Box

```haskell
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

f is a type constructor of kind `* -> *`. "A box of"

`fmap g x` means "apply *g* to every *a* in the box *x* to produce a box of *b*'s"

```haskell
data Maybe a = Just a | Nothing
instance Functor Maybe where
  fmap _ Nothing  = Nothing
  fmap g (Just x) = Just (g x)
```

```haskell
data Either a b = Left a | Right b
instance Functor (Either a) where
  fmap _ (Left x)  = Left x
  fmap g (Right y) = Right (g y)
```

```haskell
data List a = Cons a (List a) | Nil
instance Functor List where
  fmap g (Cons x xs) = Cons (g x) (fmap g xs)
  fmap _ Nil         = Nil
```

IO as a Functor

Functor takes a type constructor of kind `* -> *`, which is the kind of *IO*

```
Prelude> :k IO
IO :: * -> *
```

IO does behave like a kind of box:

```
query :: IO String
query = do line <- getLine            -- getLine returns a box :: IO String
           let res = line ++ "!"      -- take line out of box from getLine
           return res                 -- put res in an IO box
```

The definition of Functor IO in the Prelude: (alternative syntax)

```
instance Functor IO where
  fmap f action = do result <- action   -- take result from the box
                     return (f result)  -- apply f; put it a box
```

Using fmap with I/O Actions

```haskell
main = do line <- getLine
          let revLine = reverse line      -- Tedious but correct
          putStrLn revLine
```

```haskell
main = do revLine <- fmap reverse getLine -- More direct
          putStrLn revLine
```

```
Prelude> fmap (++"!") getLine
foo
"foo!"
```

Functions are Functors

```
Prelude> :k (->)
(->) :: * -> * -> *        -- Like ``(+),'' (->) is a function on types
```

That is, the function type constructor -> takes two concrete types and produces a third (a function). This is the same kind as *Either*

```
Prelude> :k ((->) Int)
((->) Int) :: * -> *
```

The `((->) Int)` type constructor takes type *a* and produces functions that tranform Ints to *a*'s. fmap will apply a function that transforms the *a*'s to *b*'s.

```
instance Functor ((->) a) where
  fmap f g = \x -> f (g x) -- Wait, this is just function composition!
```

```
instance Functor ((->) a) where
  fmap = (.)               -- Much more succinct (Prelude definition)
```

Fmapping Functions: fmap f g = f . g

```
Prelude> :t fmap (*3) (+100)
fmap (*3) (+100) :: Num b => b -> b

Prelude> fmap (*3) (+100) 1
303

Prelude> (*3) `fmap` (+100) $ 1
303

Prelude> (*3) . (+100) $ 1
303

Prelude> fmap (show . (*3)) (+100) 1
"303"
```

Partially Applying *fmap*

```
Prelude> :t fmap
fmap :: Functor f => (a -> b) -> f a -> f b
```

```
Prelude> :t fmap (*3)
fmap (*3) :: (Functor f, Num b) => f b -> f b
```

"`fmap (*3)`" is a function that operates on functors of the Num type class ("functors over numbers"). The function `(*3)` has been *lifted* to functors

```
Prelude> :t fmap (replicate 3)
fmap (replicate 3) :: Functor f => f a -> f [a]
```

"`fmap (replicate 3)`" is a function over functors that generates "boxed lists"

Functor Laws

Applying the identity function does not change the functor
("fmap does not change the box"):

```
fmap id = id
```

Applying *fmap* with two functions is like applying their composition
("applying functions to the box is like applying them in the box"):

```
fmap (f . g) = fmap f . fmap g

fmap (\y -> f (g y)) x = fmap f (fmap g x) -- Equivalent
```

```haskell
data Maybe a = Just a | Nothing

{- Does Maybe follow the laws? -}

instance Functor Maybe where
  fmap _ Nothing  = Nothing
  fmap f (Just x) = Just (f x)
```

```haskell
fmap id Nothing  = Nothing              -- from the definition of fmap
fmap id (Just x) = Just (id x)          -- from the definition of fmap
                 = Just x               -- from the definition of id

(fmap f . fmap g) Nothing = fmap f (fmap g Nothing)  -- def of .
                          = fmap f Nothing           -- def of fmap
                          = Nothing                  -- def of fmap
                          = fmap (f . g) Nothing     -- def of fmap

(fmap f . fmap g) (Just x) = fmap f (fmap g (Just x))-- def of .
                           = fmap f (Just (g x))     -- def of fmap
                           = Just (f (g x))          -- def of fmap
                           = Just ((f . g) x)        -- def of .
                           = fmap (f . g) (Just x)   -- def of fmap
```

My So-Called Functor

```haskell
data CMaybe a = CNothing | CJust Int a
                  deriving Show
instance Functor CMaybe where   -- Purported
  fmap _ CNothing     = CNothing
  fmap f (CJust c x) = CJust (c+1) (f x)
```

```
*Main> fmap id CNothing
CNothing             -- OK: fmap id Nothing = id Nothing
*Main> fmap id (CJust 42 "Hello")
CJust 43 "Hello"     -- FAIL: fmap id /= id because 43 /= 42

*Main> fmap ( (+1) . (+1) ) (CJust 42 100)
CJust 43 102
*Main> (fmap (+1) . fmap (+1)) (CJust 42 100)
CJust 44 102     -- FAIL: fmap (f . g) /= fmap f . fmap g because 43 /= 44
```

Multi-Argument Functions on Functors: Applicative Functors

Functions in Hakell are Curried:

```
1 + 2 = (+) 1 2 = ((+) 1) 2 = (1+) 2 = 3
```

What if we wanted to perform 1+2 in a Functor?

```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

fmap is “apply a normal function to a functor, producing a functor”

Say we want to add 1 to 2 in the [] Functor (lists):

```
[1] + [2] = (+) [1] [2]            -- Infix to prefix
          = (fmap (+) [1]) [2]     -- fmap: apply function to functor
          = [(1+)] [2]             -- Now what?
```

We want to apply a Functor containing functions to another functor, e.g., something with the signature `[a -> b] -> [a] -> [b]`

Applicative Functors: Applying Functions in a Functor

```
infixl 4 <*>
class Functor f => Applicative f where
  pure  :: a -> f a                       -- Box something, e.g., a function
  (<*>) :: f (a -> b) -> f a -> f b       -- Apply boxed function to a box
```

```
instance Applicative Maybe where
  pure = Just                             -- Put it in a "Just" box
  Nothing <*> _ = Nothing                 -- No function to apply
  Just f <*> m  = fmap f m                -- Apply function-in-a-box f
```

```
Prelude> :t fmap (+) (Just 1)
fmap (+) (Just 1) :: Num a => Maybe (a -> a) -- Function-in-a-box

Prelude> fmap (+) (Just 1) <*> (Just 2)
Just 3
Prelude> fmap (+) Nothing <*> (Just 2)
Nothing                                   -- Nothing is a buzzkiller
```

Pure and the <$> Operator

```
Prelude> pure (-) <*> Just 10 <*> Just 4
Just 6
Prelude> pure (10-) <*> Just 4
Just 6
Prelude> (-) `fmap` (Just 10) <*> Just 4
Just 6
```

<$> is simply an infix *fmap* meant to remind you of the $ operator

```
infixl 4 <$>
(<$>) :: Functor f => (a -> b) -> f a -> f b
f <$> x = fmap f x        -- Or equivalently, f `fmap` x
```

So *f* <$> *x* <*> *y* <*> *z* is like *f x y z* but on applicative functors x, y, z

```
Prelude> (+) <$> [1] <*> [2]
[3]
Prelude> (,,) <$> Just "PFP" <*> Just "Rocks" <*> Just "Out"
Just ("PFP","Rocks","Out")
```

Maybe as an Applicative Functor

```
instance Functor Maybe where
  fmap _ Nothing  = Nothing
  fmap g (Just x) = Just (g x)

infixl 4 <$>
f <$> x = fmap f x
```

```
infixl 4 <*>
instance Applicative Maybe where
  pure = Just
  Nothing <*> _ = Nothing
  Just f  <*> m = fmap f m
```

```
     f <$> Just x   <*> Just y
= (  f <$> Just x ) <*> Just y    -- a <$> b <*> c = (a <$> b) <*> c
= (fmap f (Just x)) <*> Just y    -- Definition of <$>
= (      Just (f x)) <*> Just y   -- Definition of fmap Maybe
=        fmap (f x)     (Just y)  -- Definition of <*>
=        Just (f x y)             -- Definition of fmap Maybe
```

Lists are Applicative Functors

```haskell
instance Applicative [] where
  pure x = [x]                                 -- Pure makes singleton list
  fs <*> xs = [ f x | f <- fs, x <- xs ]       -- All combinations
```

`<*>` associates (evaluates) left-to-right, so the last list is iterated over first:

```
Prelude> [ (++"!"), (++"?"), (++".") ] <*> [ "Run", "GHC" ]
["Run!","GHC!","Run?","GHC?","Run.","GHC."]

Prelude> [ x+y | x <- [100,200,300], y <- [1..3] ]
[101,102,103,201,202,203,301,302,303]

Prelude> (+) <$> [100,200,300] <*> [1..3]
[101,102,103,201,202,203,301,302,303]

Prelude> pure (+) <*> [100,200,300] <*> [1..3]
[101,102,103,201,202,203,301,302,303]
```

IO is an Applicative Functor

<*> enables I/O actions to be used more like functions

```
instance Applicative IO where
  pure = return
  a <*> b = do f <- a
               x <- b
               return (f x)
```

Specialized to IO actions,

```
(<*>) :: IO (a -> b)
      -> IO a
      -> IO b
```

```
main = do
  a <- getLine
  b <- getLine
  putStrLn $ a ++ b
```

```
main :: IO ()
main = do
  a <- (++) <$> getLine <*> getLine
  putStrLn a
```

```
$ stack runhaskell af2.hs
One
Two
OneTwo
```

Function Application ((->) a) as an Applicative Functor

```
pure :: b -> ((->) a) b
     :: b -> a -> b
(<*>) :: ((->) a) (b -> c) -> ((->) a) b -> ((->) a) c
      :: (a -> b -> c) -> (a -> b) -> (a -> c)
```

The “box” is “a function that takes an *a* and returns the type in the box”

`<*>` takes `f :: a -> b -> c` and `g :: a -> b` and should produce `a -> c`.

Applying an argument `x :: a` to `f` and `g` gives `g x :: b` and `f x :: b -> c`. This means applying `g x` to `f x` gives `c`, i.e., `f x (g x) :: c`.

```
instance Applicative ((->) a) where
  pure x  = \_ -> x            -- a.k.a., const
  f <*> g = \x -> f x (g x)  -- Takes an a and uses f & g to produce a c
```

```
Prelude> :t \f g x -> f x (g x)
\f g x -> f x (g x) :: (a -> b -> c) -> (a -> b) -> a -> c
```

Functions as Applicative Functors

```
instance Applicative ((->) a) where f <*> g = \x -> f x (g x)
instance Functor     ((->) a) where fmap = (.)
f <$> x = fmap f x
```

```
Prelude> :t (+) <$> (+3) <*> (*100)
(+) <$> (+3) <*> (*100) :: Num b => b -> b -- A function on numbers
Prelude> ( (+) <$> (+3) <*> (*100) )   5
508                    -- Apply 5 to +3, apply 5 to *100, and add the results
```

Single-argument functions (+3), (*100) are the boxes (arguments are “put inside”), which are assembled with (+) into a single-argument function.

```
  (       (+) <$> (+3) <*> (*100)    ) 5
= (       ((+) . (+3)) <*> (*100)    ) 5 -- Definition of <$>
= (\x -> ((+) . (+3)) x  ((*100) x)) 5 -- Definition of <*>
=         ((+) . (+3)) 5  ((*100) 5))    -- Apply 5 to lambda expr.
=         ((+) ((+3) 5))  ((*100) 5))    -- Definition of .
=          (+)  8             500        -- Evaluate (+3) 5, (*100) 5
=          508                           -- Evaluate (+) 8 500
```

Functions as Applicative Functors

Another example: `(,,)` is the "build a 3-tuple operator"

```
Prelude> :t (,,) <$> (+3) <*> (*3) <*> (*100)
(,,) <$> (+3) <*> (*3) <*> (*100) :: Num a => a -> (a, a, a)

Prelude> ((,,)  <$>  (+3)  <*>  (*3)  <*>  (*100))   2
(5,6,200)
```

The elements of the 3-tuple:

2 + 3 = 5
2 * 3 = 6
2 * 100 = 200

Each comes from applying 2 to the three functions.

"Generate a 3-tuple by applying the argument to (+3), (*3), and (*100)"

ZipList Applicative Functors

The usual implementation of Applicative Functors on lists generates all possible combinations:

```
Prelude> [(+),(*)] <*> [1,2] <*> [10,100]
[11,101,12,102,10,100,20,200]
```

Control.Applicative provides an alternative approach with zip-like behavior:

```
newtype ZipList a = ZipList { getZipList :: [a] }
instance Applicative ZipList where
  pure x = ZipList (repeat x)    -- Infinite list of x's
  ZipList fs <*> ZipList xs = ZipList (zipWith (\f x -> f x) fs xs)
```

```
> ZipList [(+),(*)] <*> ZipList [1,2] <*> ZipList [10,100]
ZipList {getZipList = [11,200]}          -- [1 + 10, 2 * 100]
> pure (,,) <*> ZipList [1,2] <*> ZipList [3,4] <*> ZipList [5,6]
ZipList {getZipList = [(1,3,5),(2,4,6)]}
```

liftA2: Lift a Two-Argument Function to an Applicative Functor

```
class Functor f => Applicative f where
  pure   :: a -> f a
  (<*>)  :: f (a -> b) -> f a -> f b
  (<*>)  = liftA2 id        -- Default: get function from 1st arg's box

  liftA2 :: (a -> b -> c) -> f a -> f b -> f c
  liftA2 f x = (<*>) (fmap f x)      -- Default implementation
```

liftA2 takes a binary function and “lifts” it to work on boxed values, e.g.,

```
  liftA2 :: (a -> b -> c) -> (f a -> f b -> f c)
```

```
Prelude Control.Applicative> liftA2 (:) (Just 3) (Just [4])
Just [3,4]             -- Apply (:) inside the boxes, i.e., Just ((:) 3 [4])
```

```
instance Applicative ZipList where
    pure x = ZipList (repeat x)
    liftA2 f (ZipList xs) (ZipList ys) = ZipList (zipWith f xs ys)
```

Turning a list of boxes into a box containing a list

```
sequenceA1 :: Applicative f => [f a] -> f [a] -- Prelude sequenceA
sequenceA1 []     = pure []
sequenceA1 (x:xs) = (:) <$> x <*> sequenceA1 xs
```

```
*Main> sequenceA1 [Just 3, Just 2, Just 1]
Just [3,2,1]
```

Recall that f <$> Just x <*> Just y = Just ($f\ x\ y$)

```
sequenceA1 [Just 3, Just 1]
= (:) <$> Just 3 <*> sequenceA1 [Just 1]
= (:) <$> Just 3 <*> ((:) <$> Just 1 <*> sequenceA1 [])
= (:) <$> Just 3 <*> ((:) <$> Just 1 <*> pure [])
= (:) <$> Just 3 <*> ((:) <$> Just 1 <*> Just [])
= (:) <$> Just 3 <*> Just [1]
= Just [3,1]
```

SequenceA Can Also Be Implemented With a Fold

```haskell
import Control.Applicative (liftA2)

sequenceA2 :: Applicative f => [f a] -> f [a] -- Prelude sequenceA
sequenceA2 = foldr (liftA2 (:)) (pure [])
```

How do the types work out?

```
liftA2 ::  App. f ⇒ (a → b   → c  ) → f a → f  b  → f  c
(:)   ::              a → [a] → [a]
```

Passing (:) to liftA2 makes b = [a] and c = [a], so

```
liftA2 (:) :: App. f ⇒                f a → f [a] → f [a]
```

```
foldr ::                              (d   → e     → e) → e     → [d]   → e
```

Passing liftA2 (:) to foldr makes d = f a and e = f [a], so

```
foldr (liftA2 (:))  :: App. f ⇒                         f [a] → [f a] → f [a]
```

```
pure [] ::                 App. f ⇒                     f [a]
foldr (liftA2 (:)) (pure []) :: App. f ⇒                        [f a] → f [a]
```

SequenceA in Action

```
sequenceA :: Applicative f => [f a] -> f [a]
sequenceA = foldr (liftA2 (:)) (pure [])
```

"Take the items from a list of boxes to make a box with a list of items"

```
Prelude> sequenceA [Just 3, Just 2, Just 1]
Just [3,2,1]
Prelude> sequenceA [Just 3, Nothing, Just 1]
Nothing                            -- ``Nothing'' nullifies the result

Prelude> :t sequenceA [(+3), (+2), (+1)]
sequenceA [(+3), (+2), (+1)] :: Num a => a -> [a] -- Produces a list
Prelude> sequenceA [(+3), (+2), (+1)] 10
[13,12,11]                         -- Apply the argument to each function

Prelude> sequenceA [[1,2,3],[10,20]]
[[1,10],[1,20],[2,10],[2,20],[3,10],[3,20]] -- fmap on lists
```

Applicative Functor Laws

```
              pure f <*> x = fmap f x      -- <*>: apply a boxed function

             pure id <*> x = x             -- Because fmap id = id

pure (.) <*> x <*> y <*> z = x <*> (y <*> z)   -- <*> is left-to-right

        pure f <*> pure x = pure (f x)        -- Apply a boxed function

              x <*> pure y = pure ($ y) <*> x -- ($ y): “apply arg. y”
```

The *newtype* keyword: Build a New Type From an Existing Type

Say you want a version of an existing type only usable in certain contexts. *type* makes an alias with no restrictions. *newtype* is a more efficient version of *data* that only allows a single data constructor

```haskell
newtype DegF = DegF { getDegF :: Double }
newtype DegC = DegC { getDegC :: Double }

fToC :: DegF -> DegC
fToC (DegF f) = DegC $ (f - 32) * 5 / 9

cToF :: DegC -> DegF
cToF (DegC c) = DegF $ (c * 9 / 5) + 32

instance Show DegF where show (DegF f) = show f ++ "F"

instance Show DegC where show (DegC c) = show c ++ "C"
```

DegF and DegC In Action

```
*Main> fToC (DegF 32)
0.0C
*Main> fToC (DegF 98.6)
37.0C
*Main> cToF (DegC 37)
98.6F
*Main> cToF 33
    * No instance for (Num DegC) arising from the literal '33'
*Main> DegC 33 + DegC 32
    * No instance for (Num DegC) arising from a use of '+'
*Main> let t1 = DegC 33
*Main|     t2 = DegC 10 in
*Main| getDegC t1 + getDegC t2
43.0
```

Newtype vs. Data: Slightly Faster and Lazier

```
newtype DegF = DegF { getDegF :: Double }
data    DegF = DegF { getDegF :: Double } -- Same syntax
```

A *newtype* may only have a single data constructor with a single field

Compiler treats a *newtype* as the encapsulated type, so it's slightly faster

Pattern matching always succeeds for a *newtype*:

```
Prelude> data DT    = DT Bool
Prelude> newtype NT = NT Bool

Prelude> helloDT (DT _) = "hello"
Prelude> helloNT (NT _) = "hello"

Prelude> helloDT undefined
"*** Exception: Prelude.undefined
Prelude> helloNT undefined
"hello"                           -- Just a Bool in NT's clothing
```

Data vs. Type vs. NewType

Keyword	When to use
`data`	When you need a completely new algebraic type or record, e.g., `data MyTree a = Node a (MyTree a) (MyTree a) | Leaf`
`type`	When you want a concise name for an existing type and aren't trying to restrict its use, e.g., `type String = [Char]`
`newtype`	When you're trying to restrict the use of an existing type and were otherwise going to write `data MyType = MyType` *t*

Monoids

Type classes present a common interface to types that behave similarly

A *Monoid* is a type with an associative binary operator and an identity value

E.g., * and 1 on numbers, ++ and [] on lists:

```
Prelude> 4 * 1
4   -- 1 is the identity on the right
Prelude> 1 * 4
4   -- 1 is the identity on the left
Prelude> 2 * (3 * 4)
24
Prelude> (2 * 3) * 4
24   -- * is associative
Prelude> 2 * 3
6
Prelude> 3 * 2
6   -- * happens to be commutative
```

```
Prelude> "hello" ++ []
"hello"   -- [] is the right identity
Prelude> [] ++ "hello"
"hello"   -- [] is the left identity
Prelude> "a" ++ ("bc" ++ "de")
"abcde"
Prelude> ("a" ++ "bc") ++ "de"
"abcde"   -- ++ is associative
Prelude> "a" ++ "b"
"ab"
Prelude> "b" ++ "a"
"ba"      -- ++ is not commutative
```

The Monoid Type Class

```
class Monoid m where
  mempty  :: a                        -- The identity value
  mappend :: m -> m -> m              -- The associative binary operator

  mconcat :: [m] -> m                 -- Apply the binary operator to a list
  mconcat = foldr mappend mempty      -- Default implementation
```

Lists are Monoids:

```
instance Monoid [a] where
  mempty  = []
  mappend = (++)
```

```
Prelude> mempty :: [a]
[]
Prelude> "hello " `mappend` "world!"
"hello world!"
Prelude> mconcat ["hello ","pfp ","world!"]
"hello pfp world!"
```

*, 1 and +, 0 Can Each Make a Monoid

newtype lets us build distinct Monoids for each

In Data.Monoid,

```
newtype Product a = Product { getProduct :: a }
  deriving (Eq, Ord, Read, Show, Bounded)

instance Num a => Monoid (Product a) where
  mempty = Product 1
  Product x `mappend` Product y = Product (x * y)
```

```
newtype Sum a = Sum { getSum :: a }
  deriving (Eq, Ord, Read, Show, Bounded)

instance Num a => Monoid (Sum a) where
  mempty = Sum 0
  Sum x `mappend` Sum y = Sum (x + y)
```

Product and Sum In Action

```
Prelude Data.Monoid> mempty :: Sum Int
Sum {getSum = 0}
Prelude Data.Monoid> mempty :: Product Int
Product {getProduct = 1}

Prelude Data.Monoid> Sum 3 `mappend` Sum 4
Sum {getSum = 7}
Prelude Data.Monoid> Product 3 `mappend` Product 4
Product {getProduct = 12}

Prelude Data.Monoid> mconcat [Sum 1, Sum 10, Sum 100]
Sum {getSum = 111}
Prelude Data.Monoid> mconcat [Product 10, Product 3, Product 5]
Product {getProduct = 150}
```

The Any (||, False) and All (&&, True) Monoids

In Data.Monoid,

```haskell
newtype Any = Any { getAny :: Bool }
  deriving (Eq, Ord, Read, Show, Bounded)

instance Monoid Any where
  mempty = Any False
  Any x `mappend` Any y = Any (x || y)
```

```haskell
newtype All = All { getAll :: Bool }
  deriving (Eq, Ord, Read, Show, Bounded)

instance Monoid All where
  mempty = All True
  All x `mappend` All y = All (x && y)
```

Any and All

```
Prelude Data.Monoid> mempty :: Any
Any {getAny = False}
Prelude Data.Monoid> mempty :: All
All {getAll = True}

Prelude Data.Monoid> getAny $ Any True `mappend` Any False
True
Prelude Data.Monoid> getAll $ All True `mappend` All False
False

Prelude Data.Monoid> mconcat [Any True, Any False, Any True]
Any {getAny = True}
Prelude Data.Monoid> mconcat [All True, All True, All False]
All {getAll = False}
```

Yes, *any* and *all* are easier to use

Ordering as a Monoid

```
data Ordering = LT | EQ | GT
```

In Data.Monoid,

```
instance Monoid Ordering where
  mempty = EQ
  LT `mappend` _ = LT
  EQ `mappend` y = y
  GT `mappend` _ = GT
```

Application: an *lcomp* for strings ordered by length then alphabetically, e.g.,

```
lcomp :: String -> String -> Ordering

"b"       `lcomp` "aaaa"    = LT -- b is shorter
"bbbbb"   `lcomp` "a"       = GT -- bbbbb is longer
"avenger" `lcomp` "avenged" = LT -- Same length: r is after d
```

lcomp

```haskell
lcomp :: String -> String -> Ordering
lcomp x y = case length x `compare` length y of
            LT -> LT
            GT -> GT
            EQ -> x `compare` y
```

A little too operational; *mappend* is exactly what we want

```haskell
lcomp :: String -> String -> Ordering
lcomp x y = (length x `compare` length y) `mappend`
            (x `compare` y)
```

Maybe the Monoid

```haskell
instance Monoid a => Monoid (Maybe a) where
  mempty = Nothing
  Nothing `mappend` m       = m
  m       `mappend` Nothing = m
  Just m1 `mappend` Just m2 = Just (m1 `mappend` m2)
```

```
Prelude> Nothing `mappend` Just "pfp"
Just "pfp"
Prelude> Just "fun" `mappend` Nothing
Just "fun"

Prelude> :m +Data.Monoid
Prelude Data.Monoid> Just (Sum 3) `mappend` Just (Sum 4)
Just (Sum {getSum = 7})
```

The Foldable Type Class

What I taught you:

```
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr _ a []     = a
foldr f a (x:xs) = f x (foldr f a xs)
```

How it's actually defined (Data.Foldable):

```
foldr :: Foldable t => (a -> b -> b) -> b -> t a -> b
```

```haskell
class Foldable t where
  {-# MINIMAL foldMap | foldr #-}
  foldr, foldr' :: (a -> b -> b) -> b -> t a -> b
  foldr1        :: (a -> a -> a) -> t a -> a
  foldl, foldl' :: (b -> a -> b) -> b -> t a -> b
  foldl1        :: (a -> a -> a) -> t a -> a
  fold          :: Monoid m => t m -> m                -- with mappend
  foldMap       :: Monoid m => (a -> m) -> t a -> m
  toList        :: t a -> [a]
  null          :: t a -> Bool
  length        :: t a -> Int
  elem          :: Eq a => a -> t a -> Bool
  maximum       :: Ord a => t a -> a
  minimum       :: Ord a => t a -> a
  sum           :: Num a => t a -> a
  product       :: Num a => t a -> a
```

Instance of Foldable for [] is just the usual list functions

```haskell
data Tree a = Node a (Tree a) (Tree a) | Nil deriving (Eq, Read)

instance Foldable Tree where
  foldMap _ Nil          = mempty
  foldMap f (Node x l r) = foldMap f l `mappend`
                           f x         `mappend`
                           foldMap f r
```

```haskell
> foldl (+) 0 (fromList [5,3,1,2,4,6,7] :: Tree Int)
28                          -- folding the tree
> getSum $ foldMap Sum $ fromList [5,3,1,2,4,6,7]
28                          -- The Sum Monoid's mappend is +
> getAny $ foldMap (\x -> Any $ x == 'w') $ fromList "brown"
True                        -- Any's mappend is ||
> getAny $ foldMap (Any . (=='w')) $ fromList "brown"
True                        -- More concise
> foldMap (\x -> [x]) $ fromList [5,3,1,2,4,6,7]
[1,2,3,4,5,6,7]             -- List's mappend is ++
```