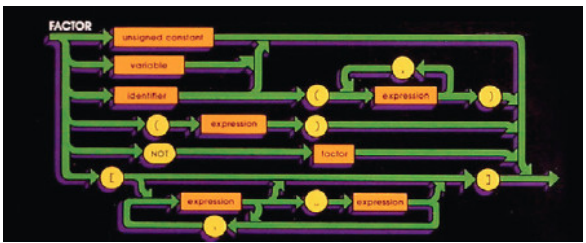
Context-Free Grammars

Stephen A. Edwards

Columbia University

Spring 2023



Context-Free Grammar

$G = (V, \Sigma, P, s)$ is a *context free grammar*

- V is a set of variables or *non-terminals*
- Σ is an alphabet of *terminals* or tokens where $V \cap \Sigma = \emptyset$
- $P \subset V \times (V \cup \Sigma)^*$ are a set of *production rules*
- $s \in V$ is the *start symbol*

$$V = \{expr, dig\}$$

$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$

$$s = expr$$

$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$
$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$expr$ Start symbol

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$

$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$\underline{expr}$	Start symbol
$expr\ \texttt{+}\ expr$	$expr \longrightarrow expr\ \texttt{+}\ expr$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$
$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$\underline{expr}$	Start symbol
$expr\ \texttt{+}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{+}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ expr$	$expr \longrightarrow expr\ \texttt{*}\ expr$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$
$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$\underline{expr}$	Start symbol
$expr\ \texttt{+}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{+}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{*}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ dig$	$expr \longrightarrow dig$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$

$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$\underline{expr}$	Start symbol
$expr\ \texttt{+}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{+}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{*}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{dig}$	$expr \longrightarrow dig$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{3}$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$

$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$\underline{expr}$	Start symbol
$expr\ \texttt{+}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{+}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{*}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{dig}$	$expr \longrightarrow dig$
$expr\ \texttt{+}\ \underline{expr}\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{3}$
$expr\ \texttt{+}\ dig\ \texttt{*}\ \texttt{3}$	$expr \longrightarrow dig$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$
$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$\underline{expr}$	Start symbol
$expr\ \texttt{+}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{+}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{*}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{dig}$	$expr \longrightarrow dig$
$expr\ \texttt{+}\ \underline{expr}\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{3}$
$expr\ \texttt{+}\ \underline{dig}\ \texttt{*}\ \texttt{3}$	$expr \longrightarrow dig$
$expr\ \texttt{+}\ \texttt{2}\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{2}$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$

$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$\underline{expr}$	Start symbol
$expr\ \texttt{+}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{+}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{*}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{dig}$	$expr \longrightarrow dig$
$expr\ \texttt{+}\ \underline{expr}\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{3}$
$expr\ \texttt{+}\ \underline{dig}\ \texttt{*}\ \texttt{3}$	$expr \longrightarrow dig$
$\underline{expr}\ \texttt{+}\ \texttt{2}\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{2}$
$\underline{dig}\ \texttt{+}\ \texttt{2}\ \texttt{*}\ \texttt{3}$	$expr \longrightarrow dig$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$
$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

The Language of a Grammar

A grammar derives a string by starting from the start symbol and replacing a non-terminal according to a production rule.

The set of all strings of terminals are exactly those that can be derived.

Rightmost Derivation; Expand Underlined

$\underline{expr}$	Start symbol
$expr\ \texttt{+}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{+}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{expr}$	$expr \longrightarrow expr\ \texttt{*}\ expr$
$expr\ \texttt{+}\ expr\ \texttt{*}\ \underline{dig}$	$expr \longrightarrow dig$
$expr\ \texttt{+}\ \underline{expr}\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{3}$
$expr\ \texttt{+}\ \underline{dig}\ \texttt{*}\ \texttt{3}$	$expr \longrightarrow dig$
$\underline{expr}\ \texttt{+}\ \texttt{2}\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{2}$
$\underline{dig}\ \texttt{+}\ \texttt{2}\ \texttt{*}\ \texttt{3}$	$expr \longrightarrow dig$
$\texttt{1}\ \texttt{+}\ \texttt{2}\ \texttt{*}\ \texttt{3}$	$dig \longrightarrow \texttt{1}$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$

$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

Backus-Naur Notation

⟨adding operator⟩ ::= +|−
⟨multiplying operator⟩ ::= ×|/|÷
⟨primary⟩ ::= ⟨unsigned number⟩|⟨variable⟩|
⟨function designator⟩|(⟨arithmetic expression⟩)
⟨factor⟩ ::= ⟨primary⟩|⟨factor⟩↑⟨primary⟩
⟨term⟩ ::= ⟨factor⟩|⟨term⟩⟨multiplying opreator⟩⟨factor⟩
⟨simple arithmetic expression⟩ ::= ⟨term⟩|
⟨adding operator⟩⟨term⟩|⟨simple arithmetic expression⟩
⟨adding operator⟩⟨term⟩

Peter Naur, John Backus et al., Report on the Algorithmic Language ALGOL 60.

$$V = \{expr, dig\}$$

$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$

$$s = expr$$

$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr\ \texttt{+}\ expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr\ \texttt{-}\ expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr\ \texttt{*}\ expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

Backus-Naur Notation

$$\begin{array}{lll} expr & ::= & expr\ \texttt{+}\ expr \\ & | & expr\ \texttt{-}\ expr \\ & | & expr\ \texttt{*}\ expr \\ & | & dig \\ dig & ::= & \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \texttt{3} \mid \texttt{4} \mid \texttt{5} \mid \texttt{6} \mid \texttt{7} \mid \texttt{8} \mid \texttt{9} \end{array}$$

$$V = \{expr, dig\}$$
$$\Sigma = \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{0}, \texttt{1}, \texttt{2}, \texttt{3}, \texttt{4}, \texttt{5}, \texttt{6}, \texttt{7}, \texttt{8}, \texttt{9}\}$$
$$s = expr$$

$$P = \left\{ \begin{array}{llll} expr \longrightarrow expr \texttt{ + } expr & dig \longrightarrow \texttt{0} & dig \longrightarrow \texttt{5} \\ expr \longrightarrow expr \texttt{ - } expr & dig \longrightarrow \texttt{1} & dig \longrightarrow \texttt{6} \\ expr \longrightarrow expr \texttt{ * } expr & dig \longrightarrow \texttt{2} & dig \longrightarrow \texttt{7} \\ expr \longrightarrow dig & dig \longrightarrow \texttt{3} & dig \longrightarrow \texttt{8} \\ & dig \longrightarrow \texttt{4} & dig \longrightarrow \texttt{9} \end{array} \right\}$$

Backus-Naur Notation

$$\begin{array}{lll} expr & ::= & expr \texttt{ + } expr \\ & & expr \texttt{ - } expr \\ & & expr \texttt{ * } expr \\ & & dig \\ dig & ::= & \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9} \end{array}$$

dig ::= **0** | **1** | … | **9**

expr ::= *dig* | *expr* **+** *expr* | *expr* **-** *expr* | *expr* ***** *expr*

$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero}$ $\frac{}{\texttt{1}\ \text{dig}}\ \text{one}$ … $\frac{}{\texttt{9}\ \text{dig}}\ \text{nine}$

dig ::= **0** | **1** | … | **9**

$expr$ ::= dig | $expr$ **+** $expr$ | $expr$ **-** $expr$ | $expr$ ***** $expr$

$$\frac{}{\mathbf{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\mathbf{1}\ \text{dig}}\ \text{one} \qquad \dots \qquad \frac{}{\mathbf{9}\ \text{dig}}\ \text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\ \text{dig}$$

$dig ::= \mathtt{0} \mid \mathtt{1} \mid \ldots \mid \mathtt{9}$

$expr ::= dig \mid expr\ \mathtt{+}\ expr \mid expr\ \mathtt{-}\ expr \mid expr\ \mathtt{*}\ expr$

$$\frac{}{\mathtt{0}\ \text{dig}}\,\text{zero} \qquad \frac{}{\mathtt{1}\ \text{dig}}\,\text{one} \qquad \ldots \qquad \frac{}{\mathtt{9}\ \text{dig}}\,\text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\,\text{dig} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathtt{+}\ e_2\ \text{expr}}\,\text{plus}$$

$dig ::= \mathtt{0} \mid \mathtt{1} \mid \ldots \mid \mathtt{9}$

$expr ::= dig \mid expr\ \mathtt{+}\ expr \mid expr\ \mathtt{-}\ expr \mid expr\ \mathtt{*}\ expr$

$$\frac{}{\mathtt{0}\ \text{dig}}\,\text{zero} \qquad \frac{}{\mathtt{1}\ \text{dig}}\,\text{one} \qquad \ldots \qquad \frac{}{\mathtt{9}\ \text{dig}}\,\text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\,\text{dig} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathtt{+}\ e_2\ \text{expr}}\,\text{plus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathtt{-}\ e_2\ \text{expr}}\,\text{minus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathtt{*}\ e_2\ \text{expr}}\,\text{times}$$

dig ::= **0** | **1** | … | **9**

$expr$ ::= dig | $expr$ **+** $expr$ | $expr$ **-** $expr$ | $expr$ ***** $expr$

$$\frac{}{\mathbf{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\mathbf{1}\ \text{dig}}\ \text{one} \qquad \ldots \qquad \frac{}{\mathbf{9}\ \text{dig}}\ \text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\ \text{dig} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{+}\ e_2\ \text{expr}}\ \text{plus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{-}\ e_2\ \text{expr}}\ \text{minus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{*}\ e_2\ \text{expr}}\ \text{times}$$

1 + 2 * 3 expr

dig ::= **0** | **1** | … | **9**

$expr$ ::= dig | $expr$ **+** $expr$ | $expr$ **-** $expr$ | $expr$ ***** $expr$

$$\frac{}{\mathbf{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\mathbf{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\mathbf{9}\ \text{dig}}\ \text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\ \text{dig} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{+}\ e_2\ \text{expr}}\ \text{plus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{-}\ e_2\ \text{expr}}\ \text{minus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{*}\ e_2\ \text{expr}}\ \text{times}$$

$$\frac{\mathbf{1}\ \text{expr} \qquad \mathbf{2}\ \mathbf{*}\ \mathbf{3}\ \text{expr}}{\mathbf{1}\ \mathbf{+}\ \mathbf{2}\ \mathbf{*}\ \mathbf{3}\ \text{expr}}\ \text{plus}$$

dig ::= **0** | **1** | … | **9**

$expr$ ::= dig | $expr$ **+** $expr$ | $expr$ **-** $expr$ | $expr$ ***** $expr$

$$\frac{}{\texttt{0}\ \text{dig}}\,\text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\,\text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\,\text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\,\text{dig} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \texttt{+}\ e_2\ \text{expr}}\,\text{plus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \texttt{-}\ e_2\ \text{expr}}\,\text{minus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \texttt{*}\ e_2\ \text{expr}}\,\text{times}$$

$$\dfrac{\dfrac{\texttt{1}\ \text{dig}}{\texttt{1}\ \text{expr}}\,\text{dig} \qquad \texttt{2 * 3}\ \text{expr}}{\texttt{1 + 2 * 3}\ \text{expr}}\,\text{plus}$$

dig ::= **0** | **1** | … | **9**

$expr$ ::= dig | $expr$ **+** $expr$ | $expr$ **-** $expr$ | $expr$ ***** $expr$

$$\frac{}{\mathbf{0}\ \text{dig}}\,\text{zero} \qquad \frac{}{\mathbf{1}\ \text{dig}}\,\text{one} \qquad \cdots \qquad \frac{}{\mathbf{9}\ \text{dig}}\,\text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\,\text{dig} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{+}\ e_2\ \text{expr}}\,\text{plus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{-}\ e_2\ \text{expr}}\,\text{minus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{*}\ e_2\ \text{expr}}\,\text{times}$$

$$\dfrac{\dfrac{\dfrac{}{\mathbf{1}\ \text{dig}}\,\text{one}}{\mathbf{1}\ \text{expr}}\,\text{dig} \qquad \mathbf{2\ *\ 3}\ \text{expr}}{\mathbf{1\ +\ 2\ *\ 3}\ \text{expr}}\,\text{plus}$$

dig ::= **0** | **1** | … | **9**

$expr$ ::= dig | $expr$ **+** $expr$ | $expr$ **-** $expr$ | $expr$ ***** $expr$

$$\frac{}{\texttt{0 dig}}\,\text{zero} \qquad \frac{}{\texttt{1 dig}}\,\text{one} \qquad \cdots \qquad \frac{}{\texttt{9 dig}}\,\text{nine}$$

$$\frac{d\ \texttt{dig}}{d\ \texttt{expr}}\,\text{dig} \qquad \frac{e_1\ \texttt{expr} \quad e_2\ \texttt{expr}}{e_1\ \texttt{+}\ e_2\ \texttt{expr}}\,\text{plus} \qquad \frac{e_1\ \texttt{expr} \quad e_2\ \texttt{expr}}{e_1\ \texttt{-}\ e_2\ \texttt{expr}}\,\text{minus} \qquad \frac{e_1\ \texttt{expr} \quad e_2\ \texttt{expr}}{e_1\ \texttt{*}\ e_2\ \texttt{expr}}\,\text{times}$$

$$\dfrac{\dfrac{\dfrac{}{\texttt{1 dig}}\,\text{one}}{\texttt{1 expr}}\,\text{dig} \quad \dfrac{\dfrac{\dfrac{}{\texttt{2 dig}}\,\text{two}}{\texttt{2 expr}}\,\text{dig} \quad \dfrac{\dfrac{}{\texttt{3 dig}}\,\text{three}}{\texttt{3 expr}}\,\text{dig}}{\texttt{2 * 3 expr}}\,\text{times}}{\texttt{1 + 2 * 3 expr}}\,\text{plus}$$

dig ::= **0** | **1** | … | **9**

$expr$ ::= dig | $expr$ **+** $expr$ | $expr$ **-** $expr$ | $expr$ ***** $expr$

$$\frac{}{\mathbf{0}\ \text{dig}}\,\text{zero} \qquad \frac{}{\mathbf{1}\ \text{dig}}\,\text{one} \qquad \cdots \qquad \frac{}{\mathbf{9}\ \text{dig}}\,\text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\,\text{dig} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{+}\ e_2\ \text{expr}}\,\text{plus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{-}\ e_2\ \text{expr}}\,\text{minus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{*}\ e_2\ \text{expr}}\,\text{times}$$

$$\dfrac{\dfrac{\dfrac{}{\mathbf{1}\ \text{dig}}\,\text{one}}{\mathbf{1}\ \text{expr}}\,\text{dig} \quad \dfrac{\dfrac{\dfrac{}{\mathbf{2}\ \text{dig}}\,\text{two}}{\mathbf{2}\ \text{expr}}\,\text{dig} \quad \dfrac{\dfrac{}{\mathbf{3}\ \text{dig}}\,\text{three}}{\mathbf{3}\ \text{expr}}\,\text{dig}}{\mathbf{2}\ \mathbf{*}\ \mathbf{3}\ \text{expr}}\,\text{times}}{\mathbf{1}\ \mathbf{+}\ \mathbf{2}\ \mathbf{*}\ \mathbf{3}\ \text{expr}}\,\text{plus} \qquad \mathbf{1}\ \mathbf{+}\ \mathbf{2}\ \mathbf{*}\ \mathbf{3}\ \text{expr}$$

dig ::= **0** | **1** | … | **9**

$expr$ ::= dig | $expr$ **+** $expr$ | $expr$ **-** $expr$ | $expr$ ***** $expr$

$$\frac{}{\mathbf{0}\ \text{dig}}\,\text{zero} \qquad \frac{}{\mathbf{1}\ \text{dig}}\,\text{one} \qquad \cdots \qquad \frac{}{\mathbf{9}\ \text{dig}}\,\text{nine}$$

$$\frac{d\ \text{dig}}{d\ \text{expr}}\,\text{dig} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{+}\ e_2\ \text{expr}}\,\text{plus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{-}\ e_2\ \text{expr}}\,\text{minus} \qquad \frac{e_1\ \text{expr} \quad e_2\ \text{expr}}{e_1\ \mathbf{*}\ e_2\ \text{expr}}\,\text{times}$$

$$\dfrac{\dfrac{\dfrac{}{\mathbf{1}\ \text{dig}}\,\text{one}}{\mathbf{1}\ \text{expr}}\,\text{dig} \quad \dfrac{\dfrac{\dfrac{}{\mathbf{2}\ \text{dig}}\,\text{two}}{\mathbf{2}\ \text{expr}}\,\text{dig} \quad \dfrac{\dfrac{}{\mathbf{3}\ \text{dig}}\,\text{three}}{\mathbf{3}\ \text{expr}}\,\text{dig}}{\mathbf{2\ *\ 3}\ \text{expr}}\,\text{times}}{\mathbf{1\ +\ 2\ *\ 3}\ \text{expr}}\,\text{plus} \qquad \dfrac{\mathbf{1\ +\ 2}\ \text{expr} \quad \mathbf{3}\ \text{expr}}{\mathbf{1\ +\ 2\ *\ 3}\ \text{expr}}\,\text{times}$$

$dig ::= \mathtt{0} \mid \mathtt{1} \mid \ldots \mid \mathtt{9}$

$expr ::= dig \mid expr\ \mathtt{+}\ expr \mid expr\ \mathtt{-}\ expr \mid expr\ \mathtt{*}\ expr$

$$\frac{}{\mathtt{0}\ \mathsf{dig}}\,\mathsf{zero} \qquad \frac{}{\mathtt{1}\ \mathsf{dig}}\,\mathsf{one} \qquad \ldots \qquad \frac{}{\mathtt{9}\ \mathsf{dig}}\,\mathsf{nine}$$

$$\frac{d\ \mathsf{dig}}{d\ \mathsf{expr}}\,\mathsf{dig} \qquad \frac{e_1\ \mathsf{expr} \quad e_2\ \mathsf{expr}}{e_1\ \mathtt{+}\ e_2\ \mathsf{expr}}\,\mathsf{plus} \qquad \frac{e_1\ \mathsf{expr} \quad e_2\ \mathsf{expr}}{e_1\ \mathtt{-}\ e_2\ \mathsf{expr}}\,\mathsf{minus} \qquad \frac{e_1\ \mathsf{expr} \quad e_2\ \mathsf{expr}}{e_1\ \mathtt{*}\ e_2\ \mathsf{expr}}\,\mathsf{times}$$

$$\dfrac{\dfrac{\dfrac{}{\mathtt{1}\ \mathsf{dig}}\,\mathsf{one}}{\mathtt{1}\ \mathsf{expr}}\,\mathsf{dig} \qquad \dfrac{\dfrac{}{\mathtt{2}\ \mathsf{dig}}\,\mathsf{two}}{\mathtt{2}\ \mathsf{expr}}\,\mathsf{dig} \qquad \dfrac{\dfrac{}{\mathtt{3}\ \mathsf{dig}}\,\mathsf{three}}{\mathtt{3}\ \mathsf{expr}}\,\mathsf{dig}}{\mathtt{2\ *\ 3}\ \mathsf{expr}}\,\mathsf{times}}{\mathtt{1\ +\ 2\ *\ 3}\ \mathsf{expr}}\,\mathsf{plus}$$

$$\dfrac{\dfrac{\dfrac{}{\mathtt{1}\ \mathsf{dig}}\,\mathsf{one}}{\mathtt{1}\ \mathsf{expr}}\,\mathsf{dig} \qquad \dfrac{\dfrac{}{\mathtt{2}\ \mathsf{dig}}\,\mathsf{two}}{\mathtt{2}\ \mathsf{expr}}\,\mathsf{dig}}{\mathtt{1\ +\ 2}\ \mathsf{expr}}\,\mathsf{plus} \qquad \dfrac{\dfrac{}{\mathtt{3}\ \mathsf{dig}}\,\mathsf{three}}{\mathtt{3}\ \mathsf{expr}}\,\mathsf{dig}}{\mathtt{1\ +\ 2\ *\ 3}\ \mathsf{expr}}\,\mathsf{times}$$

$dig ::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}$

$expr ::= dig \mid expr \texttt{ + } expr \mid expr \texttt{ - } expr \mid expr \texttt{ * } expr$

$$\frac{}{\texttt{0 dig}}\,\text{zero} \qquad \frac{}{\texttt{1 dig}}\,\text{one} \qquad \ldots \qquad \frac{}{\texttt{9 dig}}\,\text{nine}$$

$$\frac{d\ \texttt{dig}}{d\ \texttt{expr}}\,\text{dig} \qquad \frac{e_1\ \texttt{expr} \quad e_2\ \texttt{expr}}{e_1\ \texttt{+}\ e_2\ \texttt{expr}}\,\text{plus} \qquad \frac{e_1\ \texttt{expr} \quad e_2\ \texttt{expr}}{e_1\ \texttt{-}\ e_2\ \texttt{expr}}\,\text{minus} \qquad \frac{e_1\ \texttt{expr} \quad e_2\ \texttt{expr}}{e_1\ \texttt{*}\ e_2\ \texttt{expr}}\,\text{times}$$

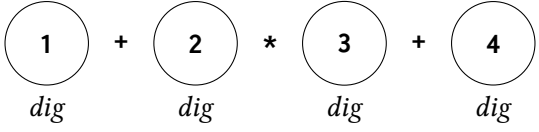
$$\dfrac{\dfrac{\dfrac{}{\texttt{1 dig}}\,\text{one}}{\texttt{1 expr}}\,\text{dig} \qquad \dfrac{\dfrac{\dfrac{}{\texttt{2 dig}}\,\text{two}}{\texttt{2 expr}}\,\text{dig} \qquad \dfrac{\dfrac{}{\texttt{3 dig}}\,\text{three}}{\texttt{3 expr}}\,\text{dig}}{\texttt{2 * 3 expr}}\,\text{times}}{\texttt{1 + 2 * 3 expr}}\,\text{plus}$$

$$\dfrac{\dfrac{\dfrac{}{\texttt{1 dig}}\,\text{one}}{\texttt{1 expr}}\,\text{dig} \qquad \dfrac{\dfrac{}{\texttt{2 dig}}\,\text{two}}{\texttt{2 expr}}\,\text{dig}}{\texttt{1 + 2 expr}}\,\text{plus} \qquad \dfrac{\dfrac{}{\texttt{3 dig}}\,\text{three}}{\texttt{3 expr}}\,\text{dig}}{\texttt{1 + 2 * 3 expr}}\,\text{times}$$

Resolving Ambiguity Through Restructuring

expr ::= *expr* **+** *expr*
| *expr* **-** *expr*
| *expr* ***** *expr*
| *dig*

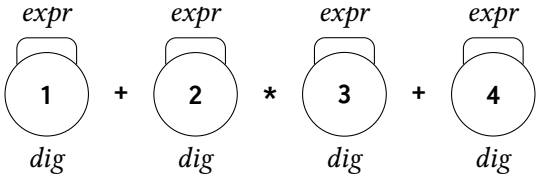
dig ::= **0** | **1** | ... | **9**



Resolving Ambiguity Through Restructuring

expr ::= *expr* **+** *expr*
| *expr* **-** *expr*
| *expr* ***** *expr*
| *dig*

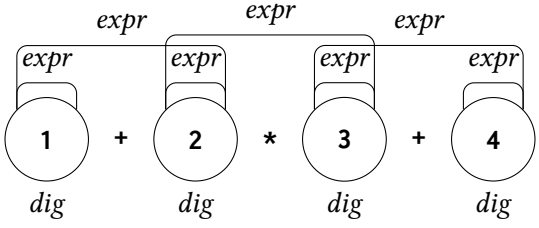
dig ::= **0** | **1** | ... | **9**



Resolving Ambiguity Through Restructuring

expr ::= *expr* **+** *expr*
| *expr* **−** *expr*
| *expr* ***** *expr*
| *dig*

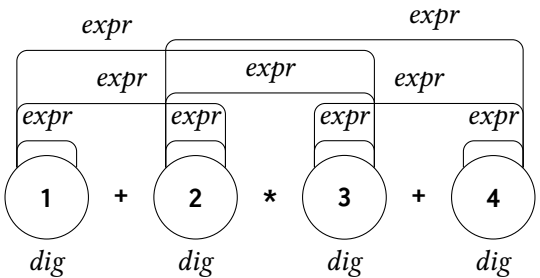
dig ::= **0** | **1** | … | **9**



Resolving Ambiguity Through Restructuring

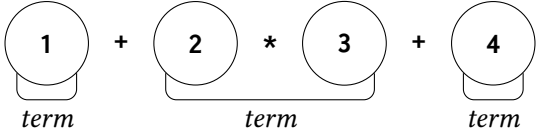
expr ::= *expr* **+** *expr*
| *expr* **-** *expr*
| *expr* ***** *expr*
| *dig*

dig ::= **0** | **1** | ... | **9**



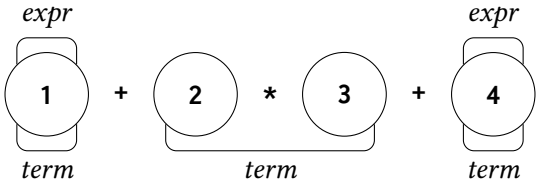
Resolving Ambiguity Through Restructuring

$$
\begin{aligned}
expr ::=\ & expr\ \texttt{+}\ expr \\
\mid\ & expr\ \texttt{-}\ expr \\
\mid\ & expr\ \texttt{*}\ expr \\
\mid\ & dig \\
dig\ ::=\ & \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
\qquad
\begin{aligned}
expr ::=\ & expr\ \texttt{+}\ expr \\
\mid\ & expr\ \texttt{-}\ expr \\
\mid\ & term \\
term ::=\ & term\ \texttt{*}\ term \\
\mid\ & dig \\
dig\ ::=\ & \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$



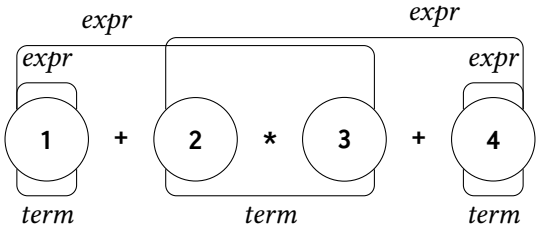
Resolving Ambiguity Through Restructuring

$$
\begin{aligned}
expr ::=\ & expr\ \texttt{+}\ expr \\
\mid\ & expr\ \texttt{-}\ expr \\
\mid\ & expr\ \texttt{*}\ expr \\
\mid\ & dig \\
dig\ ::=\ & \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
\qquad
\begin{aligned}
expr ::=\ & expr\ \texttt{+}\ expr \\
\mid\ & expr\ \texttt{-}\ expr \\
\mid\ & term \\
term ::=\ & term\ \texttt{*}\ term \\
\mid\ & dig \\
dig\ ::=\ & \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$



Resolving Ambiguity Through Restructuring

$$
\begin{aligned}
expr &::= expr \texttt{ + } expr \\
&\quad\ \ expr \texttt{ - } expr \\
&\quad\ \ expr \texttt{ * } expr \\
&\quad\ \ dig \\
dig &::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
\qquad
\begin{aligned}
expr &::= expr \texttt{ + } expr \\
&\quad\ \ expr \texttt{ - } expr \\
&\quad\ \ term \\
term &::= term \texttt{ * } term \\
&\quad\ \ dig \\
dig &::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$



Resolving Ambiguity Through Restructuring

expr ::= *expr* **+** *expr*
| *expr* **−** *expr*
| *expr* ***** *expr*
| *dig*

dig ::= **0** | **1** | … | **9**

expr ::= *expr* **+** *expr*
| *expr* **−** *expr*
| *term*

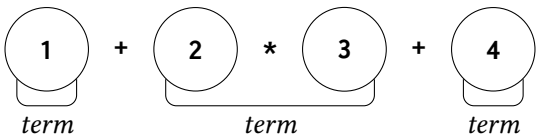
term ::= *term* ***** *term*
| *dig*

dig ::= **0** | **1** | … | **9**

expr ::= *expr* **+** *term*
| *expr* **−** *term*
| *term*

term ::= *term* ***** *dig*
| *dig*

dig ::= **0** | **1** | … | **9**



Resolving Ambiguity Through Restructuring

expr ::= *expr* **+** *expr*
| *expr* **−** *expr*
| *expr* ***** *expr*
| *dig*

dig ::= **0** | **1** | … | **9**

expr ::= *expr* **+** *expr*
| *expr* **−** *expr*
| *term*

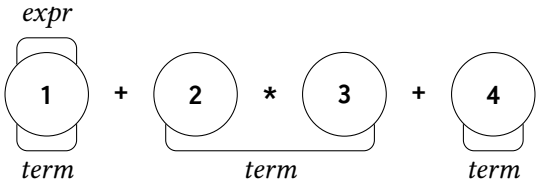
term ::= *term* ***** *term*
| *dig*

dig ::= **0** | **1** | … | **9**

expr ::= *expr* **+** *term*
| *expr* **−** *term*
| *term*

term ::= *term* ***** *dig*
| *dig*

dig ::= **0** | **1** | … | **9**

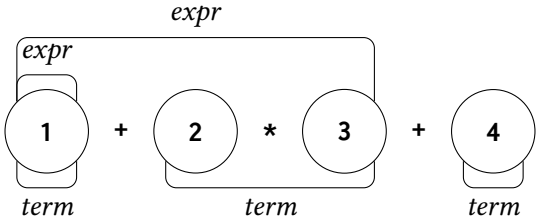


Resolving Ambiguity Through Restructuring

$$
\begin{aligned}
expr &::= expr \texttt{ + } expr \\
&\quad\ \ expr \texttt{ - } expr \\
&\quad\ \ expr \texttt{ * } expr \\
&\quad\ \ dig \\
dig &::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$

$$
\begin{aligned}
expr &::= expr \texttt{ + } expr \\
&\quad\ \ expr \texttt{ - } expr \\
&\quad\ \ term \\
term &::= term \texttt{ * } term \\
&\quad\ \ dig \\
dig &::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$

$$
\begin{aligned}
expr &::= expr \texttt{ + } term \\
&\quad\ \ expr \texttt{ - } term \\
&\quad\ \ term \\
term &::= term \texttt{ * } dig \\
&\quad\ \ dig \\
dig &::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$

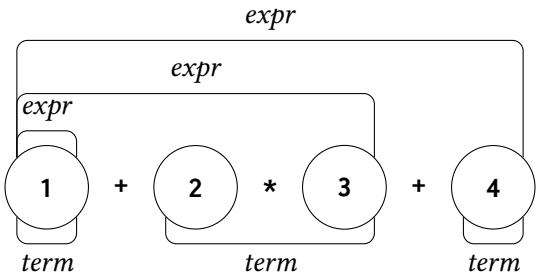


Resolving Ambiguity Through Restructuring

$$
\begin{aligned}
expr &::= expr \texttt{ + } expr \\
&\quad\ \ expr \texttt{ - } expr \\
&\quad\ \ expr \texttt{ * } expr \\
&\quad\ \ dig \\
dig &::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$

$$
\begin{aligned}
expr &::= expr \texttt{ + } expr \\
&\quad\ \ expr \texttt{ - } expr \\
&\quad\ \ term \\
term &::= term \texttt{ * } term \\
&\quad\ \ dig \\
dig &::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$

$$
\begin{aligned}
expr &::= expr \texttt{ + } term \\
&\quad\ \ expr \texttt{ - } term \\
&\quad\ \ term \\
term &::= term \texttt{ * } dig \\
&\quad\ \ dig \\
dig &::= \texttt{0} \mid \texttt{1} \mid \ldots \mid \texttt{9}
\end{aligned}
$$



dig ::= **0** | **1** | ... | **9**

term ::= *term* ***** *dig* | *dig*

expr ::= *expr* **+** *term* | *expr* **−** *term* | *term*

0 dig zero **1** dig one ... **9** dig nine

dig ::= **0** | **1** | … | **9**

$term$ ::= $term$ ***** dig | dig

$expr$ ::= $expr$ **+** $term$ | $expr$ **-** $term$ | $term$

$$\frac{}{\mathbf{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\mathbf{1}\ \text{dig}}\ \text{one} \qquad \ldots \qquad \frac{}{\mathbf{9}\ \text{dig}}\ \text{nine}$$

$$\frac{t\ \text{term} \quad d\ \text{dig}}{t\ \mathbf{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$dig ::= \mathtt{0} \mid \mathtt{1} \mid \ldots \mid \mathtt{9}$$

$$term ::= term\ \mathtt{*}\ dig \mid dig$$

$$expr ::= expr\ \mathtt{+}\ term \mid expr\ \mathtt{-}\ term \mid term$$

$$\frac{}{\mathtt{0}\ \mathsf{dig}}\ \text{zero} \qquad \frac{}{\mathtt{1}\ \mathsf{dig}}\ \text{one} \qquad \ldots \qquad \frac{}{\mathtt{9}\ \mathsf{dig}}\ \text{nine} \qquad \frac{t\ \mathsf{term} \quad d\ \mathsf{dig}}{t\ \mathtt{*}\ d\ \mathsf{term}}\ \text{term-times} \qquad \frac{d\ \mathsf{dig}}{d\ \mathsf{term}}\ \text{term-dig}$$

$$\frac{e\ \mathsf{expr} \quad t\ \mathsf{term}}{e\ \mathtt{+}\ t\ \mathsf{expr}}\ \text{expr-plus} \qquad \frac{e\ \mathsf{expr} \quad t\ \mathsf{term}}{e\ \mathtt{-}\ t\ \mathsf{expr}}\ \text{expr-minus} \qquad \frac{t\ \mathsf{term}}{t\ \mathsf{expr}}\ \text{expr-term}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\texttt{1 + 2 * 3 + 4}\ \text{expr}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\frac{\texttt{1 + 2 * 3}\ \text{expr} \qquad \texttt{4}\ \text{term}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\texttt{1}\ \text{expr} \qquad \texttt{2 * 3}\ \text{term}}{\texttt{1 + 2 * 3}\ \text{expr}}\ \text{expr-plus} \qquad \texttt{4}\ \text{term}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\texttt{1}\ \text{term}}{\texttt{1}\ \text{expr}}\ \text{expr-term} \qquad \texttt{2 * 3}\ \text{term}}{\texttt{1 + 2 * 3}\ \text{expr}}\ \text{expr-plus} \qquad \texttt{4}\ \text{term}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\texttt{1}\ \text{dig}}{\texttt{1}\ \text{term}}\ \text{term-dig}}{\texttt{1}\ \text{expr}}\ \text{expr-term} \qquad \texttt{2 * 3}\ \text{term}}{\texttt{1 + 2 * 3}\ \text{expr}}\ \text{expr-plus} \qquad \texttt{4}\ \text{term}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{1}\ \text{dig}}\ \text{one}}{\texttt{1}\ \text{term}}\ \text{term-dig}}{\texttt{1}\ \text{expr}}\ \text{expr-term} \qquad \texttt{2 * 3}\ \text{term}}{\texttt{1 + 2 * 3}\ \text{expr}}\ \text{expr-plus} \qquad \texttt{4}\ \text{term}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\mathtt{0}\ \mathsf{dig}}\ \text{zero} \qquad \frac{}{\mathtt{1}\ \mathsf{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\mathtt{9}\ \mathsf{dig}}\ \text{nine} \qquad \frac{t\ \mathsf{term} \quad d\ \mathsf{dig}}{t\ \mathtt{*}\ d\ \mathsf{term}}\ \text{term-times} \qquad \frac{d\ \mathsf{dig}}{d\ \mathsf{term}}\ \text{term-dig}$$

$$\frac{e\ \mathsf{expr} \quad t\ \mathsf{term}}{e\ \mathtt{+}\ t\ \mathsf{expr}}\ \text{expr-plus} \qquad \frac{e\ \mathsf{expr} \quad t\ \mathsf{term}}{e\ \mathtt{-}\ t\ \mathsf{expr}}\ \text{expr-minus} \qquad \frac{t\ \mathsf{term}}{t\ \mathsf{expr}}\ \text{expr-term}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\mathtt{1}\ \mathsf{dig}}\ \text{one}}{\mathtt{1}\ \mathsf{term}}\ \text{term-dig}}{\mathtt{1}\ \mathsf{expr}}\ \text{expr-term} \qquad \cfrac{\mathtt{2}\ \mathsf{term} \qquad \mathtt{3}\ \mathsf{dig}}{\mathtt{2}\ \mathtt{*}\ \mathtt{3}\ \mathsf{term}}\ \text{term-times}}{\mathtt{1}\ \mathtt{+}\ \mathtt{2}\ \mathtt{*}\ \mathtt{3}\ \mathsf{expr}}\ \text{expr-plus} \qquad \mathtt{4}\ \mathsf{term}}{\mathtt{1}\ \mathtt{+}\ \mathtt{2}\ \mathtt{*}\ \mathtt{3}\ \mathtt{+}\ \mathtt{4}\ \mathsf{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{1}\ \text{dig}}\ \text{one}}{\texttt{1}\ \text{term}}\ \text{term-dig}}{\texttt{1}\ \text{expr}}\ \text{expr-term} \qquad \dfrac{\dfrac{\texttt{2}\ \text{dig}}{\texttt{2}\ \text{term}}\ \text{term-dig} \qquad \texttt{3}\ \text{dig}}{\texttt{2 * 3}\ \text{term}}\ \text{term-times}}{\texttt{1 + 2 * 3}\ \text{expr}}\ \text{expr-plus} \qquad \texttt{4}\ \text{term}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\dfrac{}{\texttt{1}\ \text{dig}}\ \text{one}}{\texttt{1}\ \text{term}}\ \text{term-dig}}{\texttt{1}\ \text{expr}}\ \text{expr-term} \qquad \dfrac{\dfrac{\dfrac{}{\texttt{2}\ \text{dig}}\ \text{two}}{\texttt{2}\ \text{term}}\ \text{term-dig} \quad \texttt{3}\ \text{dig}}{\texttt{2 * 3}\ \text{term}}\ \text{term-times}$$

$$\dfrac{\texttt{1 + 2 * 3}\ \text{expr} \qquad \texttt{4}\ \text{term}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{1}\ \text{dig}}\ \text{one}}{\texttt{1}\ \text{term}}\ \text{term-dig}}{\texttt{1}\ \text{expr}}\ \text{expr-term} \qquad \dfrac{\dfrac{\dfrac{}{\texttt{2}\ \text{dig}}\ \text{two}}{\texttt{2}\ \text{term}}\ \text{term-dig} \qquad \dfrac{}{\texttt{3}\ \text{dig}}\ \text{three}}{\texttt{2 * 3}\ \text{term}}\ \text{term-times}}{\texttt{1 + 2 * 3}\ \text{expr}}\ \text{expr-plus} \qquad \texttt{4}\ \text{term}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{1}\ \text{dig}}\ \text{one}}{\texttt{1}\ \text{term}}\ \text{term-dig}}{\texttt{1}\ \text{expr}}\ \text{expr-term} \qquad \dfrac{\dfrac{\dfrac{}{\texttt{2}\ \text{dig}}\ \text{two}}{\texttt{2}\ \text{term}}\ \text{term-dig} \qquad \dfrac{}{\texttt{3}\ \text{dig}}\ \text{three}}{\texttt{2 * 3}\ \text{term}}\ \text{term-times}}{\texttt{1 + 2 * 3}\ \text{expr}}\ \text{expr-plus} \qquad \dfrac{\dfrac{\texttt{4}\ \text{dig}}{\texttt{4}\ \text{term}}\ \text{term-dig}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \cdots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

$$\dfrac{\dfrac{\dfrac{}{\texttt{1}\ \text{dig}}\ \text{one}}{\texttt{1}\ \text{term}}\ \text{term-dig}}{\texttt{1}\ \text{expr}}\ \text{expr-term} \qquad \dfrac{\dfrac{\dfrac{}{\texttt{2}\ \text{dig}}\ \text{two}}{\texttt{2}\ \text{term}}\ \text{term-dig} \qquad \dfrac{}{\texttt{3}\ \text{dig}}\ \text{three}}{\texttt{2 * 3}\ \text{term}}\ \text{term-times}$$

$$\dfrac{\dfrac{\texttt{1}\ \text{expr} \qquad \texttt{2 * 3}\ \text{term}}{\texttt{1 + 2 * 3}\ \text{expr}}\ \text{expr-plus} \qquad \dfrac{\dfrac{}{\texttt{4}\ \text{dig}}\ \text{four}}{\texttt{4}\ \text{term}}\ \text{term-dig}}{\texttt{1 + 2 * 3 + 4}\ \text{expr}}\ \text{expr-plus}$$

$$\frac{}{\texttt{0}\ \text{dig}}\ \text{zero} \qquad \frac{}{\texttt{1}\ \text{dig}}\ \text{one} \qquad \ldots \qquad \frac{}{\texttt{9}\ \text{dig}}\ \text{nine} \qquad \frac{t\ \text{term} \quad d\ \text{dig}}{t\ \texttt{*}\ d\ \text{term}}\ \text{term-times} \qquad \frac{d\ \text{dig}}{d\ \text{term}}\ \text{term-dig}$$

$$\frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{+}\ t\ \text{expr}}\ \text{expr-plus} \qquad \frac{e\ \text{expr} \quad t\ \text{term}}{e\ \texttt{-}\ t\ \text{expr}}\ \text{expr-minus} \qquad \frac{t\ \text{term}}{t\ \text{expr}}\ \text{expr-term}$$

```
data Token = Num Int | Plus | Minus | Times

term :: [Token] -> Bool
term [Num n] = True
term tks     = case reverse tks of
               Num n : Times : tks' -> term (reverse tks')
               _                    -> False

expr :: [Token] -> Bool
expr tks = case reverse tks of
           -- a term followed by + and an expr
           -- a term followed by - and an expr
```

Left Recursion
Is Really Annoying

Reversing the Recursion

dig ::= **0** | **1** | … | **9**

term ::= *dig* ***** *term* | *dig*

expr ::= *term* **+** *expr* | *term* **−** *expr* | *term*

Judgments: $d \xrightarrow{\text{dig}} d'$ $t \xrightarrow{\text{term}} t'$ $e \xrightarrow{\text{expr}} e'$ "parses to"

Variables: *d* *t* *e*

Symbols: **0** **1** **2**…**9** **+** **−** ***** add sub lit mul (,) 0 1 … 9

Reversing the Recursion

dig ::= **0** | **1** | … | **9**

term ::= *dig* ***** *term* | *dig*

expr ::= *term* **+** *expr* | *term* **−** *expr* | *term*

Judgments: $d \xrightarrow{\text{dig}} d'$ $t \xrightarrow{\text{term}} t'$ $e \xrightarrow{\text{expr}} e'$ "parses to"

Variables: *d* *t* *e*

Symbols: **0** **1** **2**…**9** **+** **−** ***** add sub lit mul (,) 0 1 … 9

Reversing the Recursion

dig ::= **0** | **1** | … | **9**

term ::= *dig* ***** *term* | *dig*

expr ::= *term* **+** *expr* | *term* **−** *expr* | *term*

$$\frac{}{\texttt{0} \xrightarrow{\text{dig}} \text{lit}(0)}\ \text{zero} \quad \cdots \quad \frac{}{\texttt{9} \xrightarrow{\text{dig}} \text{lit}(9)}\ \text{nine}$$

Judgments: $d \xrightarrow{\text{dig}} d'$ $t \xrightarrow{\text{term}} t'$ $e \xrightarrow{\text{expr}} e'$ "parses to"

Variables: d t e

Symbols: **0** **1** **2**…**9** **+** **−** ***** add sub lit mul (,) 0 1 … 9

Reversing the Recursion

dig ::= **0** | **1** | … | **9**

term ::= *dig* ***** *term* | *dig*

expr ::= *term* **+** *expr* | *term* **−** *expr* | *term*

$$\frac{}{\mathbf{0} \xrightarrow{\text{dig}} \text{lit}(0)}\ \text{zero} \quad \cdots \quad \frac{}{\mathbf{9} \xrightarrow{\text{dig}} \text{lit}(9)}\ \text{nine} \qquad \frac{d \xrightarrow{\text{dig}} d' \quad t \xrightarrow{\text{term}} t'}{d \mathbin{*} t \xrightarrow{\text{term}} \text{mul}(d', t')}\ \text{term-times} \qquad \frac{d \xrightarrow{\text{dig}} d'}{d \xrightarrow{\text{term}} d'}\ \text{term-dig}$$

Judgments: $d \xrightarrow{\text{dig}} d'$ $t \xrightarrow{\text{term}} t'$ $e \xrightarrow{\text{expr}} e'$ "parses to"

Variables: d t e

Symbols: **0** **1** **2**…**9** **+** **−** ***** add sub lit mul (,) 0 1 … 9

Reversing the Recursion

dig ::= **0** | **1** | … | **9**

term ::= *dig* ***** *term* | *dig*

expr ::= *term* **+** *expr* | *term* **−** *expr* | *term*

$$\frac{}{\mathbf{0} \xrightarrow{\text{dig}} \text{lit}(0)}\ \text{zero} \quad \cdots \quad \frac{}{\mathbf{9} \xrightarrow{\text{dig}} \text{lit}(9)}\ \text{nine} \quad \frac{d \xrightarrow{\text{dig}} d' \quad t \xrightarrow{\text{term}} t'}{d * t \xrightarrow{\text{term}} \text{mul}(d', t')}\ \text{term-times} \quad \frac{d \xrightarrow{\text{dig}} d'}{d \xrightarrow{\text{term}} d'}\ \text{term-dig}$$

$$\frac{t \xrightarrow{\text{term}} t' \quad e \xrightarrow{\text{expr}} e'}{t + e \xrightarrow{\text{expr}} \text{add}(t', e')}\ \text{expr-plus} \quad \frac{t \xrightarrow{\text{term}} t' \quad e \xrightarrow{\text{expr}} e'}{t - e \xrightarrow{\text{expr}} \text{sub}(t', e')}\ \text{expr-minus} \quad \frac{t \xrightarrow{\text{term}} t'}{t \xrightarrow{\text{expr}} t'}\ \text{expr-term}$$

Judgments: $d \xrightarrow{\text{dig}} d'$ $t \xrightarrow{\text{term}} t'$ $e \xrightarrow{\text{expr}} e'$ "parses to"

Variables: d t e

Symbols: **0** **1** **2**…**9** **+** **−** ***** add sub lit mul (,) 0 1 … 9

```
data AST = Lit Int | Add AST AST | Sub AST AST | Mul AST AST

dig :: [Token] -> (AST, [Token])
dig (Num n : t0) = (Lit n, t0)
dig _            = error "expected a digit"

term :: [Token] -> (AST, [Token])
term t0 = let (d, t1) = dig t0 in
          case t1 of
          Times : t2 -> let (t, t3) = term t2 in (Mul d t, t3)
          _          -> (d, t1)

expr :: [Token] -> (AST, [Token])
expr tks = let (t, t1) = term tks in
           case t1 of
           Plus : t2  -> let (e, t3) = expr t2 in (Add t e, t3)
           Minus : t2 -> let (e, t3) = expr t2 in (Sub t e, t3)
           _          -> (t, t1)
```

```haskell
import Control.Monad.State

peek :: State [Token] Token
peek = do tks <- get ; case tks of t : _ -> return t
                                   []    -> return EOF

eat :: State [Token] ()
eat = do tks <- get ; case tks of _ : t -> put t
                                  []    -> return ()

dig :: State [Token] AST
dig = do d <- peek
         case d of Num n -> do eat ; return (Lit n)
                   _     -> error "expected a digit"

term :: State [Token] AST
term = do d <- dig ; op <- peek
          case op of Times -> do eat ; t <- term ; return $ Mul d t
                     _     -> return d

expr :: State [Token] AST
expr = do t <- term ; op <- peek
          case op of Plus  -> do eat ; e <- expr ; return $ Add t e
                     Minus -> do eat ; e <- expr ; return $ Sub t e
                     _     -> return t
```

Test Cases

```
1                   1

1 * 2               (1 * 2)

1 * 2 * 3           (1 * (2 * 3))

1 * 2 + 3           ((1 * 2) + 3)

1 + 2 + 3           (1 + (2 + 3))

1 - 2 - 3           (1 - (2 - 3))

1 + 2 * 3 - 4       (1 + ((2 * 3) - 4))

1 * 2 * 3 - 1 * 2 - 1 - 3 + 4 * 5
                    ((1 * (2 * 3)) - ((1 * 2) - (1 - (3 + (4 * 5)))))
```

Test Cases

```
1                 1

1 * 2             (1 * 2)

1 * 2 * 3         (1 * (2 * 3))

1 * 2 + 3         ((1 * 2) + 3)

1 + 2 + 3         (1 + (2 + 3))

1 - 2 - 3         (1 - (2 - 3))

1 + 2 * 3 - 4     (1 + ((2 * 3) - 4))

1 * 2 * 3 - 1 * 2 - 1 - 3 + 4 * 5
                  ((1 * (2 * 3)) - ((1 * 2) - (1 - (3 + (4 * 5)))))
```

$$\frac{t \xrightarrow{\text{term}} t' \quad e \xrightarrow{\text{expr}} e'}{t - e \xrightarrow{\text{expr}} \text{sub}(t', e')}\ \text{expr-minus}$$

Left Factoring

dig ::= **0** | **1** | ... | **9**

$term$ ::= $dig\ termt$

$termt$ ::= ***** $dig\ termt$ | ϵ

$expr$ ::= $term\ exprt$

$exprt$ ::= **+** $term\ exprt$ | **-** $term\ exprt$ | ϵ

Left Factoring

dig ::= **0** | **1** | … | **9**

$term$ ::= $dig\ termt$

$termt$ ::= ***** $dig\ termt$ | ϵ

$expr$ ::= $term\ exprt$

$exprt$ ::= **+** $term\ exprt$ | **-** $term\ exprt$ | ϵ

$$\frac{}{\mathbf{0} \xrightarrow{\text{dig}} \text{lit}(0)}\ \text{zero} \quad \cdots \quad \frac{}{\mathbf{9} \xrightarrow{\text{dig}} \text{lit}(9)}\ \text{nine}$$

Left Factoring

$dig \;::=\; \mathtt{0} \mid \mathtt{1} \mid \ldots \mid \mathtt{9}$

$term \;::=\; dig\ termt$

$termt \;::=\; \mathtt{*}\ dig\ termt \mid \epsilon$

$expr \;::=\; term\ exprt$

$exprt \;::=\; \mathtt{+}\ term\ exprt \mid \mathtt{-}\ term\ exprt \mid \epsilon$

$$\frac{}{\mathtt{0} \xrightarrow{\text{dig}} \text{lit}(0)}\ \text{zero} \quad \cdots \quad \frac{}{\mathtt{9} \xrightarrow{\text{dig}} \text{lit}(9)}\ \text{nine}$$

$$\frac{d \xrightarrow{\text{dig}} d' \qquad t \xrightarrow[d']{\text{termt}} t'}{d\ t \xrightarrow{\text{term}} t'}\ \text{term} \qquad \frac{}{\xrightarrow[t']{\text{termt}} t'}\ \text{termt}$$

Left Factoring

$$dig \;::=\; \mathbf{0} \mid \mathbf{1} \mid \ldots \mid \mathbf{9}$$

$$term \;::=\; dig\ termt$$

$$termt \;::=\; \mathbf{*}\ dig\ termt \mid \epsilon$$

$$expr \;::=\; term\ exprt$$

$$exprt \;::=\; \mathbf{+}\ term\ exprt \mid \mathbf{-}\ term\ exprt \mid \epsilon$$

$$\frac{}{\mathbf{0} \xrightarrow{\text{dig}} \text{lit}(0)}\ \text{zero} \quad \cdots \quad \frac{}{\mathbf{9} \xrightarrow{\text{dig}} \text{lit}(9)}\ \text{nine}$$

$$\frac{d \xrightarrow{\text{dig}} d' \quad t \xrightarrow[d']{\text{termt}} t'}{d\ t \xrightarrow{\text{term}} t'}\ \text{term}$$

$$\frac{}{\xrightarrow[t']{\text{termt}} t'}\ \text{termt}$$

$$\frac{d \xrightarrow{\text{dig}} d' \quad t \xrightarrow[\text{mul}(t',d')]{\text{termt}} t''}{\mathbf{*}\ d\ t \xrightarrow[t']{\text{termt}} t''}\ \text{mult}$$

Left Factoring

$$dig ::= \mathbf{0} \mid \mathbf{1} \mid \ldots \mid \mathbf{9}$$

$$term ::= dig\ termt$$

$$termt ::= \mathbf{*}\ dig\ termt \mid \epsilon$$

$$expr ::= term\ exprt$$

$$exprt ::= \mathbf{+}\ term\ exprt \mid \mathbf{-}\ term\ exprt \mid \epsilon$$

$$\frac{}{\mathbf{0} \xrightarrow{dig} \mathrm{lit}(0)}\ \text{zero} \qquad \cdots \qquad \frac{}{\mathbf{9} \xrightarrow{dig} \mathrm{lit}(9)}\ \text{nine}$$

$$\frac{}{\xrightarrow[e']{exprt} e'}\ \text{exprt}$$

$$\frac{d \xrightarrow{dig} d' \quad t \xrightarrow[d']{termt} t'}{d\ t \xrightarrow{term} t'}\ \text{term}$$

$$\frac{}{\xrightarrow[t']{termt} t'}\ \text{termt}$$

$$\frac{d \xrightarrow{dig} d' \quad t \xrightarrow[\mathrm{mul}(t',d')]{termt} t''}{\mathbf{*}\ d\ t \xrightarrow[t']{termt} t''}\ \text{mult}$$

$$\frac{t \xrightarrow{term} t' \quad e \xrightarrow[t']{exprt} e'}{t\ e \xrightarrow{expr} e'}\ \text{expr}$$

$$\frac{t \xrightarrow{term} t' \quad e \xrightarrow[\mathrm{add}(e',t')]{exprt} e''}{\mathbf{+}\ t\ e \xrightarrow[e']{exprt} e''}\ \text{add}$$

$$\frac{t \xrightarrow{term} t' \quad e \xrightarrow[\mathrm{sub}(e',t')]{exprt} e''}{\mathbf{-}\ t\ e \xrightarrow[e']{exprt} e''}\ \text{sub}$$

$$\frac{d \xrightarrow{\text{dig}} d' \quad t \xrightarrow[d']{\text{termt}} t'}{d\ t \xrightarrow{\text{term}} t'}\ \text{term} \qquad \frac{d \xrightarrow{\text{dig}} d' \quad t \xrightarrow[\text{mul}(t',d')]{\text{termt}} t''}{\ast\ d\ t \xrightarrow[t']{\text{termt}} t''}\ \text{mult} \qquad \frac{}{\xrightarrow[t']{\text{termt}} t'}\ \text{termt}$$

```
term :: State [Token] AST
term = dig >>= termt

termt :: AST -> State [Token] AST
termt t = do op <- peek
             case op of Times -> do eat ; d <- dig ; termt $ Mul t d
                        _     ->  return t

expr :: State [Token] AST
expr = term >>= exprt

exprt :: AST -> State [Token] AST
exprt e = do op <- peek
             case op of Plus  -> do eat ; t <- term ; exprt $ Add e t
                        Minus -> do eat ; t <- term ; exprt $ Sub e t
                        _     -> return e
```

Working Test Cases

```
1                   1

1 * 2               (1 * 2)

1 * 2 * 3           ((1 * 2) * 3)

1 * 2 + 3           ((1 * 2) + 3)

1 + 2 + 3           ((1 + 2) + 3) OK

1 - 2 - 3           ((1 - 2) - 3) OK

1 + 2 * 3 - 4       ((1 + (2 * 3)) - 4)

1 * 2 * 3 - 1 * 2 - 1 - 3 + 4 * 5
                    ((((((1 * 2) * 3) - (1 * 2)) - 1) - 3) + (4 * 5)) OK
```