
1. 20 points

(a) Plot the Fourier Transform $X(\omega)$ of a signal $x \in \text{ContSignals}$ whose total energy is 2 and such that $X(\omega) = 0$ for $|\omega - 2\pi| > \pi$.

(b) Now find the time-domain signal $x$ by taking the inverse FT of $X$.

**Answer**  Take $X$ as shown in Figure 1. By Parseval’s relation its energy is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \times 2\pi \times 2 = 2.$$ 

Recall that

$$y(t) = \frac{\sin(\pi t)}{\pi t} \overset{FT}{\rightarrow} Y(\omega) = 1, |\omega| \leq \pi, = 0, |\omega| > \pi. \quad (1)$$

Observe that $X(\omega) = 2^{1/2}Y(\omega - 2\pi)$, and so

$$x(t) = 2^{1/2}y(t)e^{2\pi t}.$$
2. 15 points Fill in the blanks.

(a) The LT of \( x(t) = tu(t) \) is \( \frac{1}{s^2} \) and its ROC is \( \text{Real}(s) > 0 \).

(b) The LT of \( x(t) = e^{-t}u(t) \) is \( \frac{1}{s+1} \) and its FT is \( \frac{1}{j\omega+1} \).

(c) The transfer function \( H(s) = \frac{s-1}{s+1} \) of an LTI system has a pole at \( \underline{\quad} \) and its impulse response is \( h(t) = \underline{\quad} \).

Answer (c) \( H(s) = \frac{s-1}{s+1} \) has a pole at \( s = -1 \). Using partial fraction expansion, \( H(s) = 1 - \frac{2}{s+1} \). Taking inverse LT gives

\[
h(t) = \delta(t) - 2e^{-t}u(t).
\]
3. **20 points** Find the solution $y(t), t \geq 0,$ of the differential equation

$$\ddot{y}(t) - 3\dot{y}(t) + 2y(t) = 0,$$

with initial condition $y(0-) = 1, \dot{y}(0-) = 1$. Check that your solution satisfies these initial conditions.

**Answer** Taking LT of the differential equation gives,

$$s^2Y(s) - sy(0-) - \dot{y}(0-) - 3sY(s) + 3y(0-) + 2Y(s) = 0,$$

so, substituting for initial conditions,

$$Y(s) = \frac{s - 2}{s^2 - 3s + 2} = \frac{1}{s - 1}.$$

Taking inverse LT gives

$$y(t) = e^t u(t).$$

Check initial conditions: $y(0) = 1,$ and $\dot{y}(0) = 1.$
4. **20 points** In Figure 1 $K$ is a real constant. Find the closed-loop transfer function $H(s)$. Use the Routh test to determine the values of $K$ for which $H$ is stable.

**Answer** The closed-loop transfer function is

$$H(s) = \frac{G}{1 + G},$$

where $G(s) = \frac{K(s+0.5)}{s} \times \frac{5}{s(s+1)}$. Substituting gives, after some simplification,

$$H(s) = \frac{5K(s + 0.5)}{s^3 + s^2 + 5Ks + 2.5K}.$$

The Routh test on the denominator $D(s) = s^3 + s^2 + 5Ks + 2.5K$ gives

<table>
<thead>
<tr>
<th>$s^3$</th>
<th>1</th>
<th>5K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>1</td>
<td>2.5K</td>
</tr>
<tr>
<td>$s^1$</td>
<td>2.5K</td>
<td>0</td>
</tr>
<tr>
<td>$s^0$</td>
<td>2.5K</td>
<td>0</td>
</tr>
</tbody>
</table>

So $H$ is stable if and only if $K > 0$. 

Figure 1: System for Problem 4
5. **25 points** In Figure 2 \( m_1 \) and \( m_2 \) are real signals with real Fourier Transforms \( M_1(f) \) and \( M_2(f) \) respectively. Suppose that \( M_1(f) = 0 \), for \( |f| > 15 \) kHz. The carrier frequency \( f_c = 100 \) kHz.

   (a) Determine the Fourier Transform \( X(f) \) of the modulated signal \( x \). Write an expression for \( |X(f)| \). What is the bandwidth of \( x \)?

   (b) Find a scheme to demodulate \( x \) and recover both signals \( m_1 \) and \( m_2 \). Prove that your scheme works.

**Answer** We have

\[
x(t) = A m_1(t) \cos 2\pi f_c t + A m_2(t) \sin 2\pi f_c t
\]

\[
= \frac{A}{2} m_1(t) [e^{j2\pi f_c t} + e^{-j2\pi f_c t}] + \frac{A}{2} m_2(t) [e^{j2\pi f_c t} - e^{-j2\pi f_c t}]
\]

\[
\longleftrightarrow X(f) = \frac{A}{2} [M_1(f - f_c) + M_1(f + f_c)] + \frac{A}{2} [M_2(f - f_c) - M_2(f + f_c)]
\]

and since \( M_1, M_2 \) are real,

\[
|X(f)| = \frac{A}{2} \left\{ [M_1^2(f - f_c) + M_1^2(f + f_c)]^{1/2} + [M_2^2(f - f_c) + M_2^2(f + f_c)]^{1/2} \right\}^{1/2}
\]

The bandwidth of \( X \) is 30 kHz, twice the bandwidth of \( m_1 \) (and \( m_2 \)).

To demodulate, we multiply \( x \) by \( \cos 2\pi f_c t \) and pass the result through a LPF and separately multiply \( x \) by \( \sin 2\pi f_c t \) and pass the result through a LPF. Then

\[
x(t) \cos 2\pi f_c t = A m_1(t) \cos^2(2\pi f_c t) + A m_2(t) \sin(2\pi f_c t) \cos(2\pi f_c t)
\]

\[
= \frac{A}{2} m_1(t) [1 + 2 \cos(4\pi f_c t)] + \frac{A}{2} m_2(t) \sin(4\pi f_c t).
\]
If this signal is passed through a LPF with bandwidth 15 KHz, the output signal will be $\frac{1}{2}m_1(t)$, since the other signals are located near $2f_c = 200$ KHz.

Similarly, $x(t)\sin 2\pi f_c t$, passed through a LPF will give as output the signal $\frac{1}{2}m_2(t)$. 