Sample Problems for MT 2

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1 Fourier transforms and sampling

1. (a) If a voltage \( v \in ContSignals \) is placed across a 1 ohm resistor, what is the total energy dissipated in the resistor?

(b) Find a signal \( x \in ContSignals \) whose energy is 1 and that is bandlimited to 0.5 Hz, i.e. if \( X(f) \) is the FT of \( x \), then \( |X(f)| = 0 \), for \( |f| > 0.5 \). Show that the signal is bandlimited and use Parseval’s relation to prove that its energy is 1.

(c) Find a signal \( x \) that is bandlimited to 1 Hz such that if it is sampled every 0.5 second, the resulting samples are given by: \( x(0) = 1 \) and \( x(0.5n) = 0 \) for all \( n \neq 0 \).

(d) Is the signal which satisfies (1c) unique or not? Explain why.

(e) Given a sequence of sample values \( x(n), -\infty < n < \infty \), find a signal \( z(t), -\infty < t < \infty \), explicitly in terms of the sample sequence such that (1) \( z \) is bandlimited to 1 Hz and (2) \( z(0.5n) = x(n) \) for all \( n \).

2. The ideal \( T \)-sampler is the system \( Sampler_T \) that converts any signal \( x \) into the signal

\[
Sampler_T(x)(t) = y(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT).
\]

The zero-order hold is the system \( ZOH_T \) that converts \( y \) into the signal

\[
ZOH_T(y)(t) = x(nT), \ nT \leq t < (n+1)T.
\]

The linear interpolator on the other hand is the system \( LI_T \) that converts \( y \) into the signal

\[
LI_T(y)(t) = x(nT) + \frac{t-nT}{T}[x((n+1)T) - x(nT)], \ nT \leq t < (n+1)T.
\]

(a) Sketch \( x, Sampler_T(x), ZOH_T(y), ZOH_T(y) \) for some signal \( x \).
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(b) Find impulse responses $h_Z, h_L$ such that

$$ZOH_T(y) = h_Z * y, \text{ and } LI_T(y) = h_L * y.$$  

(c) What is the impulse response $h_I$ such that $h_I * y$ is the ideal interpolator.

(d) Find and compare the frequency response of the three impulse responses.

3. This problem requires use of FT properties.

(a) Give examples of real-valued, non-zero signals $x$ whose FT $X$ is (a) real-valued, and 
(b) totally imaginary.

(b) If $x$ is a real-valued signal with FT $X$, what is the inverse FT of $ReX(\omega) = 1/2[X(\omega) + X^*(\omega)]$, the real value of $X$ and of $ImX(\omega) = 1/2[X(\omega) - X^*(\omega)]$?

(c) What is the FT of the signal $x$ given by

$$x(t) = \begin{cases} 1 - |t|, & \text{if } |t| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Hint. Use the fact that $x = y * y$ where $y(t) = 0.5, |t| < 1$, and $y(t) = 0$, otherwise.

4. This problem relates the DTFT of a finite-duration signal and its DTFS. (See text, p. 311.)

(a) Suppose $x \in DiscSignals$ and $x(n) = 0$ for $n < 0$ and $n \geq M$. Let $X(\omega)$ be the $2\pi$-periodic DTFT of $x$. Let $p \geq M$ and consider the $p$-point DTFS defined by

$$X_k = \frac{1}{p} \sum_{n=0}^{M-1} x(n)e^{-jk\omega_0n}, \quad k = 0, 1, \ldots, p - 1.$$  

Show that $X_k = \frac{1}{p} X(k\omega_0)$, i.e. the DTFS coefficients are "samples" of the DTFT at equally spaced frequencies ($\omega_0 = 2\pi/p$).

(b) Suppose $x \in DiscSignals$ is given by $x(n) = 1$ for $-M \leq n \leq M$, and $x(n) = 0$, otherwise. Calculate its DTFT. Let $p = 4M$. Calculate the $p$-point DTFS defined above. Plot the magnitudes of the DTFT and DTFS on the same plot.

5. In practice the FT of a continuous-time signal of infinite duration is calculated by taking the DTFS of a finite duration of the discretized signal. This problem studies the two approximations (discretization and finite duration) involved in this calculation using as example the signal $x \in ContSignals$: $t \mapsto x(t) = e^{-|t|}$.

(a) Calculate the FT $X$. Why is it real? Use Matlab to plot $X$ over the frequency range $|\omega| \leq 5\pi$.

(b) What is the total energy in this signal? Use Parseval’s relation to figure out the energy $x$ in the frequency range $|\omega| < 2\pi$. (You may not be able to do this explicitly but you should be able to express this as an integral that could be numerically evaluated.)
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(c) Suppose $x$ is sampled every $T$ seconds to obtain

$$y_T(t) = \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT),$$

and let $Y_T$ be its FT. Then $Y_T$ consists of sums of shifted versions of $X$. Why is there aliasing in this case? Use Matlab to plot $Y_T(\omega)$ for $|\omega| \leq \pi / T$ (why is $Y_T$ periodic with period $2\pi / T$?) and for $T = 0.1, 1.0, 5.0$ sec.

(d) How small should $T$ be so that $X(\pi / T) / X(0) \leq 0.01$? (We could say that for practical purposes if $T$ is smaller than this value, there is no aliasing.)

(e) Now consider the effect of “windowing.” Let $z_{T,N}$ be the finite duration sampled signal,

$$z_{T,N}(t) = \sum_{n=-N}^{N} x(t) \delta(t - nT),$$

and let $Z_{N,T}$ be its FT. Use Matlab to plot $Z_{N,T}(\omega)$ for $T = 0.1, 1.0, 5.0$ and $N = 1, 5, 10$.

(f) Finally let $\{X_{N,T,k}, -N \leq k \leq N\}$ be the $2N + 1$-point DTFS of $\{x(nT), -N \leq n \leq N\}$. Express $X_{N,T,k}$ in terms of the $\{x(nT)\}$. Also express $X_{N,T,k}$ in terms of the FT $X_{N,T}$.

6. Spectral analysis A common problem is that you observe finite samples of a sum of sinusoids,

$$x(t) = \sum_{i=1}^{M} A_i \cos(\omega_i t),$$

and your job is to figure out $M, \{A_i\}, \{\omega_i\}$, given $x(nT), 0 \leq n \leq N - 1$. This problem concerns values of the sampling period $T$ and the number of samples $N$ required to do the job.

(a) Consider first the case of a single sinusoid, with samples

$$x_n = A \cos(\omega_0 nT), \quad 0 \leq n \leq N - 1,$$

where $T$ is the sampling period, not related to $\omega_0$. The DTFT of these samples is the $2\pi$-periodic function

$$X(\omega) = \sum_{n=0}^{N-1} x_n e^{-j\omega n} = \sum_{n=0}^{N-1} A \cos(\omega_0 nT) e^{-j\omega n}.$$  

Show that

$$X(\omega) = \frac{A}{2} e^{-j(\omega - \omega_0 T) \frac{N-1}{2}} \frac{\sin[(\omega - \omega_0 T) \frac{N}{2}]}{\sin[(\omega - \omega_0 T) \frac{1}{2}]} + \frac{A}{2} e^{-j(\omega + \omega_0 T) \frac{N-1}{2}} \frac{\sin[(\omega + \omega_0 T) \frac{N}{2}]}{\sin[(\omega + \omega_0 T) \frac{1}{2}]}.$$  

(3)
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(b) The two terms in (3) are sincs and the width of each main lobe is $\frac{4\pi}{N}$. If $N$ is large, the two sincs won’t overlap and

$$|X(\omega)| \approx \frac{A}{2} \left( \frac{\sin[(\omega - \omega_0) \frac{N}{2}]}{\sin[(\omega - \omega_0) \frac{N}{2}]} + \frac{\sin[(\omega + \omega_0) \frac{N}{2}]}{\sin[(\omega + \omega_0) \frac{N}{2}]} \right).$$

Plot this magnitude and indicate the centers of the two sincs and the width of the main lobes for the case where $A = 1$, $\omega_0 = 20\pi$, and $T = 0.01, 0.02$ and $N = 10, 50, 100$. Explain how you would estimate $A$ and $\omega_0$ in the samples (2) from your plot. What is the uncertainty in your estimate of $A, \omega_0$?

(c) Now suppose that instead of the DTFT you take a $N$-point DTFS of the samples (2). Relate the DTFS with your plot above, and explain how you would estimate $A, \omega_0$ and what is the uncertainty in your estimates?

(d) How would you estimate the $A_i, \omega_i$ from an $N$-point DTFS of (1), and what will be the uncertainty in your estimate?

2 Modulation

1. The AM transmission band is between 550 kHz and 1,600 kHz. The Federal Communications Commission or FCC allocates 20 kHz to each licensed AM station. Thus the first station is assigned the band 550-570 kHz and carrier frequency 560 kHz. The next station is assigned 570-590 kHz and carrier frequency 580 kHz. And so on. Suppose all stations use DSB-LC modulation.

(a) What is the maximum bandwidth of the message signal of each signal? Sketch what you expect the spectrum between 550 kHz and 610 kHz to look like. Carefully mark the contribution of each station.

(b) Give a block diagram design of a receiver that can be used to demodulate any station. Remember that the signal received by the antenna will contain all the stations. If your design contains filters, indicate their frequency response.

(c) Suppose we wish to receive the 590-610 kHz station using your receiver. Let $m$ be the original message signal. Explain how your receiver recovers $m$. In your explanation provide sketches of the FT of key signals in your receiver.

2. Vestigial sideband modulation of $m$ is done by passing the signal $m(t) \cos(\omega_c t)$ through a filter with frequency response $H(\omega)$ that nearly suppresses (say) the lower sideband. Demodulation is done by coherent detection (i.e. multiplication by $\cos(\omega_c t)$) followed by a low-pass filter. Show that $H$ must satisfy $H(\omega + \omega_c) + H(\omega - \omega_c) \equiv 1$ for the demodulation to work.

3. Two different signals $m_1$ and $m_2$ modulate the amplitude and phase of a carrier to obtain the modulated signal

$$x(t) = A(t) \cos[2\pi f_c t + \theta(t)]$$

where $A(t) = 1 + \mu m_1(t)$ and $\theta(t) = \phi \Delta m_2(t)$. Assume $|\mu m_1(t)| < 1$ and $|\phi \Delta m_2(t)| < 1$. $\mu, \phi$ are constants of modulation.
3

LAPLACE TRANSFORMS

(a) Use the approximation \( \cos \alpha \approx 1 \) and \( \sin \alpha \approx \alpha \) for \( |\alpha| << 1 \) to obtain an approximate expression for \( x \) and then obtain the Fourier Transform of \( x \) in terms of the Fourier Transforms \( M_1(f) \) and \( M_2(f) \) of \( m_1 \) and \( m_2 \) respectively.

(b) Propose a scheme for recovering \( m_1 \) and \( m_2 \) from \( x \). Assume that the bandwidths of \( m_1 \) and \( m_2 \) are much smaller than \( f_c \).

4. Two different signals \( m_1 \) and \( m_2 \) modulate a single carrier to obtain

\[
x(t) = m_1(t) \cos(\omega_ct) + m_2(t) \sin(\omega_c t).
\]

Assume the bandwidths of \( m_1 \) and \( m_2 \) are much smaller than \( \omega_c \).

(a) Give a block diagram of a system that recovers both \( m_1 \) and \( m_2 \) from \( x \).

(b) Express the FT of all the different signals in your system.

5. This problem shows how to calculate the bandwidth of a (broadband) PM signal when the modulating signal is periodic.

(a) Let \( m \) be a periodic square wave, with fundamental frequency \( f_m \). Suppose \( m \) is centered so that it is an even function. Show that the Fourier series representation of \( m \) leads to

\[
m(t) = \sum_{k=0}^{\infty} a_k \cos(2k\pi f_m t).
\]

(b) Follow the argument in the handout for broadband FM (where \( m \) is a pure sine wave) to calculate the FT of the modulated signal

\[
x(t) = \cos(\omega_c t + m(t)),
\]

in terms of the Fourier series representation above.

(c) If \( a_k \approx 0 \) for \( k > 3 \), what is the bandwidth of \( x \) in terms of \( f_m \)?

3 Laplace Transforms

1. The input and output voltages of an electric circuit are related by the differential equation

\[
y''(t) + 3y'(t) + 2y(t) = x(t) + \dot{x}(t)
\]

(a) Suppose \( x(t) \equiv 0 \), \( y(0^-) = 0 \), \( \dot{y}(0^-) = 1 \). Calculate \( y(t), t \geq 0 \).

(b) Find the zero-state transfer function \( H(s) \), and the zero-state impulse response \( h(t) \).

(c) If \( x(t) = \cos 2tu(t) \), find the steady-state response \( y_{ss}(t) \).

2. Determine how many roots of the polynomial

\[
p(s) = s^4 + 5s^3 + s^2 + 10s + 1
\]

have real part \( > 0, < 0, = 0 \)?