**Modulation**

Recall the representation of narrow-band (NB) signals.

**Definition** $x(t), -\infty < t < \infty$, is a real NB with carrier $f_c$ (Hz) if $|X(f)| = 0, |f - f_c| > \frac{1}{2}B_x$ and $B_x << f_c$ as shown in Figure 1.

**Theorem** Let $x$ be a NB signal and $\hat{x}$ its Hilbert transform. Then $x, \hat{x}$ can be represented as

$$\forall t, \quad x(t) = A(t) \cos[2\pi f_c t + \theta(t)]$$

$$\hat{x}(t) = A(t) \sin[2\pi f_c t + \theta(t)]$$

Moreover, the amplitude $A(t)$ and phase $\theta(t)$ can be obtained from $x, \hat{x}$ and the carrier signal as follows. First obtain the complex baseband signal $z$:

$$z(t) = [x(t) + j\hat{x}(t)]e^{-j2\pi f_c t},$$

then

$$A(t) = |z(t)|, \quad \theta(t) = \arg z(t).$$

Moreover, $|Z(f)| = 0, |f| > \frac{1}{2}B_x$. 

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**Figure 1: FT of narrow-band signal**
In the context of modulation, $A$ and $\theta$ are the modulating signals, and $x$ is the modulated (transmitted) signal with carrier $f_c$. Thus the demodulator, after receiving $x$, must first obtain $\hat{x}$, then $z$, and then recover $A$ and $\theta$.

1. What is $z$ in the following two cases where we use $f_c = 100$ Hz.
   
   (a) $x(t) = \cos(2\pi \times 99t) + \sin(2\pi \times 101t)$.
   
   (b) $x$ is given in Figure 2.

2. Repeat Problem 1 but now use $f_c = 105$ Hz.

3. Given a modulating signal $f$ that is bandlimited to $W$ Hz, a carrier $\cos(\omega_c t)$ with $\omega_c >> 2\pi W$, a nonlinearity $g$, and a band-pass filter that only passes frequencies withing $-\omega_c \pm 2\pi W$ or $\omega_c \pm 2\pi W$, we want to build a modulator that has output

   $$x(t) = A[1 + \beta f(t)]\cos(\omega_c t)$$

   by combining the components as in Figure 3. Assume $|f(t)| << 1$. Compute $\beta$ for the following nonlinearities.

   (a) $g(x) = \begin{cases} 
   x, & x < 0 \\
   0, & x \geq 0 
   \end{cases}$

   (b) $g(x) = \begin{cases} 
   x, & |x| < 1/2 \\
   0, & |x| \geq 1/2 
   \end{cases}$
4. Consider a NB signal of the form
\[ x(t) = \cos(\omega_c t + \theta(t)) \]
Assume \( X(\omega) \) is zero except where \( |\omega| - |\omega_c| < 2\pi W \). (Hint. Below you may use \( |\omega| = (j\omega)(-j\text{sgn}\omega) \).)

(a) Find \( y, z \) in the upper arrangement of Figure 4.
(b) Find \( z \) in the lower arrangement. Here the bandpass filter only passes the frequencies \( |\omega| - |\omega_c| < 2\pi W \).

5. The arrangement in Figure 5 is a scrambler (i-iv) followed by a descrambler (v-vii). The single sideband modulators generate, for an input signal \( f \), the output signal \( \phi(t) = f(t) \cos Wt - \hat{f}(t) \sin Wt \). The LPF has frequency response \( H(\omega) = 1, |\omega| \leq W, \) and \( H(\omega) = 0, \) otherwise. The signal \( m \) at input i is given by its Fourier transform \( M \) in the lower part of the figure.

(a) Give graphical representations in the frequency domain of the signals at ii-vii.
(b) Explain in one sentence in which sense the signal at iv is a scrambled version of the input signal.

6. Consider the PAM transmission system (you don’t need to know what PAM is) in which the frequency response of the channel and the LPF combined is given by
\[
H(\omega) = \begin{cases} 
\frac{1}{2}(1 + \cos \frac{\omega}{2}), & |\omega| < 2\pi \\
0, & |\omega| \geq 2\pi 
\end{cases}
\]
(a) Find the (non-causal) impulse response of this system.

(b) Let the input to the PAM system be

\[ x(t) = 5\delta(t) + \delta(t - 1) + 4\delta(t - 2). \]

Use the result of the previous part to obtain a formula for the output \( y \).

(c) What are the values of \( y(0) \), \( y(1) \), \( y(2) \), \( y(3) \)?

(d) If the clock at the receiving end is delayed by \( \tau \) seconds the sampled output is \( y(\tau) \), \( y(1 + \tau) \), \( y(2 + \tau) \), \( y(3 + \tau) \). Let \( \tau = 0.2 \). Use your calculator to compute one of these values.

7. A test channel for binary transmission has the frequency response \( H(\omega) = \frac{1}{\alpha + i\omega} \). The input to the channel is given in Figure 6. The input on the left corresponds to the binary input 01 (case A), and on the right corresponds to 11 (case B). The second binary digit is sampled at the output at \( t = 1.5 \).

(a) The output at \( t = 1.5 \) will depend on whether the first binary digit is 0 or 1, and denote it as \( y_A(1.5) \) and \( y_B(1.5) \), respectively. What is the lowest possible value of \( \alpha \) if we have the requirement

\[ \frac{y_B(1.5) - y_A(1.5)}{y_A(1.5)} \leq 0.01. \]

(b) Explain intuitively why the requirement is easy to satisfy for large values of \( \alpha \). In particular, what happens to the ratio above as \( \alpha \to \infty \).