
This problem set covers material from Chapter 9. Here is a summary of root locus results.

**Theorem** (Principle of the argument) Let \( L(s) \) be a rational function, and let \( C \) be a closed curve in the complex plane that does not go through any poles of \( L \). As \( s \) traverses \( C \) clockwise, its image \( L(s) \) traverses the close curve \( L(C) \) and

\[
\text{(\# of times } L(s) \text{ encircles the origin [-1] clockwise)} = (\text{\# of zeros of } L(s)[1 + L(s)] \text{ inside } C) - (\text{\# of poles of } L(s)[1 + L(s)] \text{ inside } C)
\]

Note. Poles of \( 1 + L(s) \) are same as poles of \( L \).

**Nyquist criterion** Let the closed-loop system transfer function be \( F(s) = L(s)/[1+L(s)] \). Assume \( L \) is proper. Then \( F \) is stable if it has no poles in the RHP. Since poles of \( F \) are the zeros of \( 1 + L \), stability of \( F \) is equivalent to the condition that there are no zeros of \( 1 + L \) in the RHP.

Let \( C \) be the contour shown in Figure 1. Then

\[
(\text{\# zeros of } 1 + L \text{ in the RHP or inside } C) = (\text{\# times } L(s) \text{ encircles -1 clockwise}) + (\text{\# poles of } L \text{ inside RHP}).
\]

Special case: If \( L \) has no poles inside RHP, then closed-loop system, \( L/(1 + L) \), is stable if and only if Nyquist plot does not encircle -1 clockwise.

**Exercises**

1. Suppose \( H(s) = P(s)/Q(s) \) is the open-loop transfer function and suppose \( KH(s)/[1 + KH(s)] \) is the closed-loop transfer function, where \( K > 0 \) is the “gain.” Show that the
closed-loop poles are the roots of the equation

\[ KP(s) + Q(s) = 0. \]

Suppose degree of \( Q \) is \( n \), degree of \( P \) is \( m \), and \( n \geq m \). Show that the closed-loop system has \( n \) poles.

(a) Suppose \( m = n \). Show that as \( K \to \infty \), the closed-loop poles approach the roots of \( P(s) = 0 \).

(b) Suppose \( m < n \). Show that as \( K \to \infty \), \( m \) closed-loop poles approach the \( m \) roots of \( P(s) = 0 \), and the remaining \( n - m \) closed-loop poles approach \( \infty \).

(c) Illustrate the previous part by plotting the root locus for \( H(s) = (s + 2)/(s^2 + 1) \).

2. In Problem 1 let

\[ H(s) = \frac{s^2 + 2s + 2}{s^2 - 4s + 5}. \]

(a) Use the Routh-Hurwitz criterion for the numerator and denominator polynomials of \( H \) to determine how many poles and zeros of \( H \) are in the right-half plane and then determine how many are in the left-half plane.

(b) Use Problem 1(a) to sketch the root locus, and show that for sufficiently large \( K \) the closed-loop system must be stable.

(c) Use your calculations in part (a) to determine for what values of \( K \) is the closed-loop system stable.

(d) Pick \( K \) so that the closed-loop system is stable. Why is the closed-loop system under-damped? What is its natural frequency? Use Matlab to plot the step response. What is the overshoot? What is the rise time?

(e) Use Matlab to obtain the Bode plot of \( H \). What are the gain and phase margins?

3. Determine if the following statements are true or false and give a brief justification of your answers.

(a) The open-loop transfer function \( H(s) = s/(s^2 + 1) \) can never be stabilized by a proportional compensator.

(b) The open-loop transfer function \( H(s) = 1/s^3 \) can never be stabilized by a PD compensator of the form \((K_P + K_Ds)\).

4. Figure 2 gives the Nyquist plot of a transfer function \( H \). As shown, \( H(0) = 10 \), and as \( \omega \to \infty \), \( |H(j\omega)| \to -3 \).

(a) If \( H(s) = P(s)/Q(s) \), what is the difference in the degrees of \( P \) and \( Q \)? Why?

(b) Sketch the Bode plot of \( H \) from its Nyquist plot.

(c) Suppose the open-loop system \( H \) is stable. For each value of \( K \) (both positive and negative) determine if the closed-loop system, \( KH/(1 + KH) \), stable?
Figure 2: Nyquist plot for Problem 4