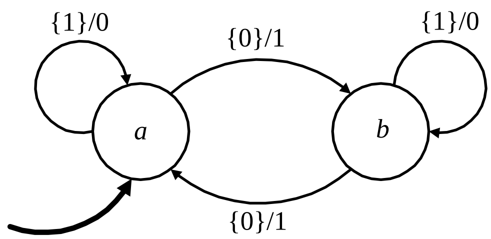
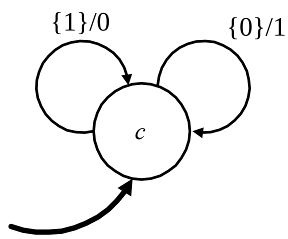
EECS 20. Final Exam Solution
December 20, 2001.

1. (a) True
   (b) True
   (c) True
   (d) False
   (e) False

2. (a) The solution is shown below:



   (b) The solution is shown below:



   (c) No, it is not well-formed. There is no fixed point.

3. (a) $$y(n) - c_1 y(n-1) - c_2 y(n-2) = x(n-2).$$
   (b) Let the state be given by (for example)
   $$s(n) = \begin{bmatrix} y(n) \\ y(n+1) \end{bmatrix}.$$
   Then
   $$A = \begin{bmatrix} 0 & 1 \\ c_2 & c_1 \end{bmatrix},\; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\; c^T = [1, 0],\; d = 0.$$
   (c) To find the frequency response, let x be such that for all n, $x(n) = e^{i\omega n}$. Then the output y is such that $y(n) = H(\omega)e^{i\omega n}$. Plug this into the difference equation and solve for $H(\omega)$ to get
   $$H(\omega) = \frac{e^{-i2\omega}}{1 - c_1 e^{-i\omega} - c_2 e^{-i2\omega}}.$$

4. (a) The first term has period 2, and the second term has period 4/3. The least common multiple is 4, so the period of the sum is 4. The fundamental frequency is therefore $f_0 = 1/4$ Hz or $\omega_0 = \pi/2$ radians/second.

   (b) $A_2 = 1$, $A_3 = 1$, $\phi_2 = -\pi/2$, $\phi_3 = 0$. All the rest of $A_k = 0$, $k \notin \{2, 3\}$, and ϕ_k for these values of k can be anything.

   (c) Write

$$x(t) = \frac{1}{2i}e^{i\pi t} - \frac{1}{2i}e^{-i\pi t} + \frac{1}{2}e^{i1.5\pi t} + \frac{1}{2}e^{-i1.5\pi t},$$

   from which we can recognize that $X_2 = 1/2i$, $X_{-2} = -1/2i$, $X_3 = 1/2$, $X_3 = 1/2$, and $X_k = 0$, $k \notin \{-3, -2, 2, 3\}$.

   (d)

$$\begin{aligned} y(n) &= x(n) = \sin(\pi n) + \cos(1.5\pi n) = \cos(1.5\pi n) \\ &= \cos((1.5\pi - 2\pi)n) = \cos(-0.5\pi n) = \cos(0.5\pi n). \end{aligned}$$

   The period is $p = 4$, so the fundamental frequency is $f_0 = 1/4$ cycles/sample, or $\omega_0 = \pi/2$ radians/sample.

   (e) Note that with $\omega_0 = \pi/2$,

$$y(n) = \cos(0.5\pi n) = \cos(\omega_0 n).$$

   Thus, $A_1 = 1$, $\phi_1 = 0$, and $A_k = 0$ for all $k \neq 1$. Note that ϕ_k for all $k \neq 1$ can be anything.

   (f) Note that

$$y(n) = \frac{1}{2}e^{i\pi n/2} + \frac{1}{2}e^{-i\pi n/2},$$

   from which we can conclude that $Y_1 = Y_{-1} = 1/2$. However, we are asked for Y_0, Y_1, Y_2, Y_3. Since Y is periodic with period p, $Y_{-1} = Y_3$, so we conclude that

$$Y_0 = 0, Y_1 = 1/2, Y_2 = 0, Y_3 = 1/2.$$

   (g) $\forall\, t \in \textit{Reals}, \quad w(t) = \cos(\pi t/2).$

   (h) $f_s > 3/2$ samples/second, or $\omega_s > 3\pi$ radians/second.

5. (a) Let the input be such that $x(t) = e^{i\omega t}$. Then

$$\begin{aligned} y(t) &= H(\omega)e^{i\omega t} \\ &= \int_{t-1}^{t+1} e^{i\omega s} ds \\ &= \frac{1}{i\omega}\int_{t-1}^{t+1} e^{i\omega s} i\omega ds \\ &= \frac{1}{i\omega}[e^{i\omega(t+1)} - e^{i\omega(t-1)}] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{i\omega} e^{i\omega t}[e^{i\omega} - e^{-i\omega}] \\
&= \frac{1}{i\omega} e^{i\omega t}[2i\sin(\omega)] \\
&= e^{i\omega t}\left[\frac{2\sin(\omega)}{\omega}\right].
\end{aligned}
$$

Solving for $H(\omega)$ yields

$$H(\omega) = \frac{2\sin(\omega)}{\omega}.$$

(b) Such input signals will have the form

$$x(t) = A\cos(\omega t + \phi)$$

for some $A \in$ *Reals*, $\phi \in$ *Reals*, and $\omega \in$ *Reals* such that

$$H(\omega) = 0.$$

Given the result of the previous part, $H(\omega) = 0$ when $\omega = k\pi$ for any $k \in$ *Integers*, $k \neq 0$.

(c) Let $x(t) = \delta(t)$, so that

$$y(t) = h(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$