1. When the inputs to a time-invariant system are: \( \forall n, \)
\[
\begin{align*}
x_1(n) &= 2\delta(n - 2) \\
x_2(n) &= \delta(n + 1)
\end{align*}
\]
where \( \delta \) is the Kronecker delta

the corresponding outputs are
\[
\begin{align*}
y_1(n) &= \delta(n - 2) + 2\delta(n - 3) \\
y_2(n) &= 2\delta(n + 1) + \delta(n)
\end{align*}
\]
respectively.

Is this system is linear? Give a proof or a counter-example.

2. Consider discrete-time systems with input and output signals \( x, y \in [\text{Integers} \rightarrow \text{Reals}] \).
Each of the following relations defines such a system. For each, indicate whether it is linear(L), time-invariant (TI), both(LTI), or neither (N). Give a proof or counter-example.

(a) \( y(n) = g(n)x(n) \)
(b) \( y(n) = e^{x(n)} \)

3. (a) An LTI system with input signal \( x \) and output signal \( y \) is described by the differential equation
\[
\ddot{y}(t) + 2\dot{y}(t) + 0.5y(t) = x(t).
\]
Suppose the input signal is \( \forall t, x(t) = e^{i\omega t} \), where \( \omega \) is fixed. What is its output signal \( y \)?

(b) Another LTI system is subject to the differential equation
\[
\ddot{y}(t) + y(t) = \dot{x}(t) + x(t)
\]
i. What is the frequency response?
ii. What is the magnitude and phase of the frequency response for \( \omega = 0.5 \)?

4. For this problem, assume discrete time everywhere. Given two LTI systems \( S \) and \( T \) suppose signal \( f \) is input into \( S \) and \( g \) into \( T \). The input and output signals are displayed in figure 1. Are the two systems identical, that is, \( S = T \)?

5. A system is described by the difference equation
\[
y(n) = x(n) + bx(n - 1) + ay(n - 1),
\]
wherein \( a, b \) are constants.

(a) Obtain the \([A, b, c^T, d]\) representation of this system by:

i. choosing the state,
ii. calculating \( A, b, c^T, d \) for your choice of state.
(b) If \( x(n - 1) = 0, y(n - 1) = 1 \), calculate the zero-input (i.e. \( x(n) = 0, n \geq 0 \)) state response.

(c) Calculate the frequency response of this system.

6. For the linear difference equation
   \[
   y(n) = 0.5y(n - 1) + x(n),
   \]

(a) Taking the state at time \( n \) to be \( s(n) = y(n - 1) \), write down the zero-input response, the zero-state impulse response \( h : Ints \rightarrow Reals \), the zero-state response, and the (full) response.

(b) Show that the zero-input response \( y_{zi} \) is a linear function of the initial state, i.e. it is of the form
   \[
   \forall n \geq 0, \quad y_{zi}(n) = a(n)s(0),
   \]
   for some constant coefficients \( a(n) \). Then show that
   \[
   \lim_{n \to \infty} y_{zi}(n) = 0
   \]

(c) Suppose \( s_0 \) is the initial state and the input is a unit step, i.e. \( x(n) = 1, n \geq 0; = 0, n < 0 \). Determine the response \( y(n), n \geq 0 \), and calculate the steady state response
   \[
   y_{ss} = \lim_{n \to \infty} y(n).
   \]

(d) Plot the input, output and the steady state value in the previous part.

(e) Calculate the frequency response \( H : Reals \rightarrow Complex \) and plot the magnitude and phase response.
(f) Suppose \( x(n) = 1, -\infty < n < \infty \). What is the output \( y(n), -\infty < n < \infty \) and compare it with \( y_{fs} \).

7. Suppose \( x \) is a continuous-time periodic signal, with period \( p \) and exponential FS representation,

\[
\forall t, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(ik\omega_0 t),
\]

in which \( \omega_0 = 2\pi / p \).

(a) Write down the formula for \( X_k \) in terms of \( x \).

(b) Consider the signal \( y \),

\[
\forall t, \quad y(t) = x(\alpha t),
\]

in which \( \alpha > 0 \) is some positive constant.

i. Show that \( y \) is periodic and find its period \( q \).

ii. Suppose \( y \) has FS representation

\[
\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k \exp(k\omega_1 t),
\]

What is \( \omega_1 \)? Determine the \( Y_k \) in terms of the \( X_k \).

8. Give an example of a nonlinear, time-invariant system \( S \) that is not memoryless. Time is discrete.

(a) Show that \( S \) is nonlinear, time-invariant, and not memoryless.

(b) Suppose \( x : \text{Ints} \to \text{Reals} \) is periodic with period \( p \). Let \( y = S(x) \). Is \( y \) periodic?

(c) Suppose \( Q \) is another discrete-time, time-invariant system. Is the cascade composition \( S \circ Q \) time-invariant? Give a proof or a counterexample.

(d) Define the system \( R \) by reversing time: \( \forall x, n, R(x)(n) = S(x)(-n) \). Is \( R \) time-invariant? Why? If \( x \) is periodic as above and \( w = R(x) \), is \( w \) periodic? Why.

9. You are given three kinds of building blocks for discrete-time systems: one-unit delay; gains; and adders.

(a) Use these building blocks to implement the system:

\[
y(n) = 0.5y(n-2) + x(n) + x(n-1).
\]

(b) Take the outputs of the delay elements as the state and give a \([A, b, d', d] \) representation of this system.

(c) You are allowed to set the output of the delay elements to any value at time \( n = 0 \). Select these values so that the output of your implementation is the solution \( y(n), n \geq 0 \) for any input \( x(n), n \geq 0 \) and initial conditions: \( y(-1) = 0.5, y(-2) = 0.8, x(-1) = 1 \). Now suppose \( x(0) = x(1) = x(2) = 0 \). Calculate \( y(0), y(1), y(2) \).
10. An integrator can be used as a building block: For any input $x : \text{Reals} \to \text{Reals}$, its output is:

$$\forall t \geq 0, \quad y(t) = y_0 + \int_0^t x(s)\,ds.$$  

The ‘initial condition’ $y(0)$ can be set.

Use integrators, gains and adders to implement the system:

$$\frac{d^2y}{dt^2}(t) - y(t) = x(t),$$

with initial condition $y(0) = 1, \dot{y}(0) = 0$.

**Hint** First convert a differential equation into an integral equation and then implement.

11. A periodic signal $x : \text{Reals} \to \text{Reals}$ is given by

$$\forall t, \quad x(t) = \left[1 + \cos(2\pi \times 10t)\right] \times \cos(2\pi \times 400t).$$

(a) What are the fundamental frequency $\omega_0$ and period $T_0$ of $x$? Calculate the Fourier Series of $x$ in the forms:

$$\forall t, \quad x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0t + \phi_k)$$

$$= \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0t}$$

Is $X_k = X^*_{-k}$?

(b) Suppose the LTI system $S$ has frequency response

$$\forall \omega, \quad H(\omega) = \begin{cases} 1, & \text{if } 2\pi \times 395 \leq |\omega| \leq 2\pi \times 405 \\ 0, & \text{otherwise} \end{cases}$$

Plot the magnitude and phase response of $H$. Repeat part 11a for $y$.

12. Give the ABCD state space representation of a discrete-time system with frequency response $H(\omega)$, where:

$$H(\omega) = \frac{2 + e^{-j\omega}}{1 - 3e^{-3j\omega}}$$

**Hint:** First find a difference equation which has the given frequency response. Then find the state space representation.

13. You are given the signal $\forall t x(t) = \cos(20\pi t) + 1 - 2 \sin(25\pi t)$ to use as input to a system with frequency response $H(\omega) = |\omega|$. Answer the following questions based on this setup.

(a) Indicate the Fourier series expansion (in cosine format) of $x$ by writing the nonzero values of $A_0$, $A_k$, and $\phi_k$ in the expansion $x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0t + \phi_k)$. 
(b) Indicate the Fourier series expansion (in complex exponential format) of \( x(t) \) by writing the nonzero values of the complex coefficients \( X_k \) in the expansion \( x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \).

(c) Give \( y \), the output of the system with input \( x \).