1. When the inputs to a time-invariant system are: \( \forall n, \)
\[
\begin{align*}
x_1(n) &= 2\delta(n - 2) \\
x_2(n) &= \delta(n + 1)
\end{align*}
\]
where \( \delta \) is the Kronecker delta

the corresponding outputs are
\[
\begin{align*}
y_1(n) &= \delta(n - 2) + 2\delta(n - 3) \\
y_2(n) &= 2\delta(n + 1) + \delta(n)
\end{align*}
\]
respectively.

Is this system is linear? Give a proof or a counter-example.

**Answer to 1** The system is not linear. From time-invariance we see that for the second pair of input and output,
\[
\begin{align*}
x_2(n - 3) &= \delta(n - 2) \\
y_2(n - 3) &= 2\delta(n - 2) + \delta(n - 3)
\end{align*}
\]
So we can rewrite the first pair of input and output as
\[
\begin{align*}
x_1(n - 3) &= 2\delta(n - 2) \\
y_1(n - 3) &= \delta(n - 2) + 2\delta(n - 3) \\
&\neq 2y_2(n - 3)
\end{align*}
\]
Therefore, the system is not linear.

2. Consider discrete-time systems with input and output signals \( x, y \in [\text{Integers} \rightarrow \text{Reals}] \). Each of the following relations defines such a system. For each, indicate whether it is linear(L), time-invariant (TI), both(LTI), or neither (N). Give a proof or counter-example.

(a) \( y(n) = g(n)x(n) \)
(b) \( y(n) = e^{x(n)} \)

**Answer to 2**
(a) The system is linear:
\[
\begin{align*}
\hat{x}(n) &= ax_1(n) + bx_2(n) \\
\hat{y}(n) &= g(n)(ax_1(n) + bx_2(n)) \\
&= ay_1(n) + by_2(n)
\end{align*}
\]
Also the system is time-varying if \( g \) is not constant (so there exist \( n, n_0 \) so that \( g(n) \neq g(n-n_0) \)):

\[
\begin{align*}
\dot{x}(n) &= x(n-n_0) \\
\dot{y}(n) &= g(n)\dot{x}(n) \\
&= g(n)x(n-n_0) \\
&\neq y(n-n_0) \\
&= g(n-n_0)x(n-n_0)
\end{align*}
\]

(b) The system is non-linear:

\[
\begin{align*}
\dot{x}(n) &= ax_1(n) + bx_2(n) \\
\dot{y}(n) &= e^{\dot{x}(n)} \\
&= e^{ax_1(n)+bx_2(n)} \\
&= (y_1(n))^a(y_2(n))^b \\
&\neq ay_1(n) + by_2(n)
\end{align*}
\]

But the system is time-invariant:

\[
\begin{align*}
\dot{x}(n) &= x(n-n_0) \\
\dot{y}(n) &= e^{\dot{x}(n)} \\
&= e^{x(n-n_0)} \\
&= y(n-n_0)
\end{align*}
\]

3. (a) An LTI system with input signal \( x \) and output signal \( y \) is described by the differential equation

\[\ddot{y}(t) + 2\dot{y}(t) + 0.5y(t) = x(t).\]

Suppose the input signal is \( \forall t, x(t) = e^{i\omega t} \), where \( \omega \) is fixed. What is its output signal \( y \)?

(b) Another LTI system is subject to the differential equation

\[\ddot{y}(t) + y(t) = \dot{x}(t) + x(t)\]

i. What is the frequency response?

ii. What is the magnitude and phase of the frequency response for \( \omega = 0.5 \)?

**Answer to 3**

(a) The output signal is \( \forall t, y(t) = H(\omega)e^{i\omega t} \). It follows that

\[-\omega^2H(\omega)e^{i\omega t} + 2i\omega H(\omega)e^{i\omega t} + 0.5H(\omega)e^{i\omega t} = e^{i\omega t},\]

thus \( H(\omega) = \frac{1}{-\omega^2 + 2i\omega + 0.5} \). Hence

\[\forall t, y(t) = \frac{1}{-\omega^2 + 2i\omega + 0.5}e^{i\omega t}\]
(b) (i) The frequency response is $H(\omega) = \frac{i\omega + 1}{\omega^2 + 1}$.

(ii) Hence

$$|H(0.5)| = \left| \frac{4}{3} + \frac{i2}{3} \right| = \frac{2\sqrt{5}}{3}, \quad \angle H(0.5) = \frac{\pi}{6}$$

4. For this problem, assume discrete time everywhere. Given two LTI systems $S$ and $T$ suppose signal $f$ is input into $S$ and $g$ into $T$. The input and output signals are displayed in figure 1. Are the two systems identical, that is, $S = T$?

![Figure 1: Signals for problem 4](image)

**Answer to 4** No. $S \neq T$ Argue by contradiction. Assume $S = T = R$, say. Observe that $f(n)$ is $(g - f)(n + 1)$. The figure below plots $R(g - f)(n) = T(g)(n) - S(f)(n)$ and $R((g - f))(n + 1) = R(f)(n) = S(f)(n)$. But the second plot is not the first plot delayed by 1.

5. A system is described by the difference equation

$$y(n) = x(n) + bx(n - 1) + ay(n - 1), \quad (1)$$

wherein $a, b$ are constants.

(a) Obtain the $[A, b, c^T, d]$ representation of this system by:

i. choosing the state,
ii. calculating $A, b, c^T, d$ for your choice of state.

(b) If $x(n - 1) = 0, y(n - 1) = 1$, calculate the zero-input (i.e. $x(n) = 0, n \geq 0$) state response.
(c) Calculate the frequency response of this system.

**Answer to 5** (a) (i) Take the state as $s(n) = [x(n-1), y(n-1)]^T$.

(ii) Writing $s(n+1) = As(n) + bx(n)$ in expanded form gives

$$s(n+1) = \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} x(n) \\ x(n) + bx(n-1) + ay(n-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ b & a \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(n),$$

from which

$$A = \begin{bmatrix} 0 & 0 \\ b & a \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and, since

$$y = [b \quad a] \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix} + x(n),$$

so $c^T = [b \quad a], \quad d = 1$.

(b) The zero-input state response is $s(n) = A^n s(0), \quad n \geq 0$. So we need to calculate $A^n$, with $A$ given in (2). By induction,

$$A^n = \begin{bmatrix} 0 & 0 \\ a^{n-1}b & a^n \end{bmatrix}$$
and since \( s(0) = [0 \ 1]^T, s(n) = [0 \ a^n]. \)

(c) To obtain the frequency response, substitute \( x(n) = e^{i\omega n}, y(n) = H(\omega)e^{i\omega n} \) in (1) and simplify to get

\[
\forall \omega, \quad H(\omega) = \frac{1 + be^{-i\omega}}{1 - ae^{i\omega}}.
\]

6. For the linear difference equation

\[ y(n) = 0.5y(n-1) + x(n), \]

(a) Taking the state at time \( n \) to be \( s(n) = y(n-1) \), write down the zero-input response, the zero-state impulse response \( h : Ints \rightarrow Reals \), the zero-state response, and the (full) response.

(b) Show that the zero-input response \( yzi \) is a linear function of the initial state, i.e. it is of the form

\[
\forall n \geq 0, \quad yzi(n) = a(n)s(0),
\]

for some constant coefficients \( a(n) \). Then show that \( \lim_{n \to \infty} yzi(n) = 0 \).

(c) Suppose \( s_0 \) is the initial state and the input is a unit step, i.e. \( x(n) = 1, n \geq 0; = 0, n < 0 \). Determine the response \( y(n), n \geq 0 \), and calculate the steady state response

\[
y_{ss} = \lim_{n \to \infty} y(n).
\]

(d) Plot the input, output and the steady state value in the previous part.

(e) Calculate the frequency response \( H : Reals \rightarrow Complex \) and plot the magnitude and phase response.

(f) Suppose \( x(n) = 1, -\infty < n < \infty \). What is the output \( y(n), -\infty < n < \infty \) and compare it with \( y_{ss} \).

**Answer to 6** (a) The \( a, b, c, d \) representation is (with \( s(n) = y(n-1) \))

\[
s(n+1) = 0.5s(n) + x(n), \quad y(n) = 0.5s(n) + x(n).
\]

The zero-input response \( (x(n) = 0, n \geq 0) \) is

\[
szi(n) = 0.5^n s(0), \quad yzi(n) = 0.5^{n+1}s(0) = 0.5^{n+1}y(-1).
\]

(3)

The zero-state impulse response is

\[
\forall n \geq 0, \quad h(n) = \begin{cases} 
  d = 1, & n = 0 \\
  ca^{n-1}b = 0.5^n, & n \geq 1
\end{cases} = 0.5^n.
\]
Figure 2: Plots for problem 6

So the full response is

\[ y(n) = 0.5^{n+1} y(-1) + \sum_{m=0}^{n} 0.5^{n-m} x(m), n \geq 0. \]  

(4)

(b) From (3) \( y_zi \) is a linear (time-varying) function of the initial state with \( a(n) = 0.3^{n+1} \). Clearly, \( y_zi(n) \rightarrow 0 \) as \( n \rightarrow \infty \).

(c) In (4) take \( x(m) = 1, m \geq 0 \) to get

\[ y(n) = 0.5^{n+1} s_0 + \sum_{m=0}^{n} m 0.5^{n-m} \times 1 \]

\[ = 0.5^{n+1} s_0 + \sum_{k=0}^{n} 0.5^{n-k} = 0.5^{n+1} s_0 = \frac{1 - 0.5^{n+1}}{1 - 0.5} \rightarrow y_{ss} = 2 \text{ as } n \rightarrow \infty \]

(d) The plots are straightforward.

(e) The frequency response is

\[ \forall \omega, \quad H(\omega) = \frac{1}{1 - 0.5 e^{i\omega}}, \]

the magnitude response is

\[ \forall \omega, \quad |H(\omega)| = \frac{1}{[1.25 - \cos(\omega)]^{1/2}}, \]

the phase response is

\[ \forall \omega, \quad \angle H(\omega) = \tan^{-1} \frac{0.5 \sin(\omega)}{1 - 0.5 \cos(\omega)}. \]

The plots in figure 2 are for \( 0 \leq \omega \leq \pi \):

(f) In this case \( x(n) \equiv e^{i0n} \), so \( y(n) \equiv H(0)e^{i0n} = 2 = y_{ss} \).
7. Suppose \( x \) is a continuous-time periodic signal, with period \( p \) and exponential FS representation,
\[
\forall t, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(i\omega_0 t),
\]
in which \( \omega_0 = 2\pi/p \).

(a) Write down the formula for \( X_k \) in terms of \( x \).

(b) Consider the signal \( y \),
\[
\forall t, \quad y(t) = x(\alpha t),
\]
in which \( \alpha > 0 \) is some positive constant.

i. Show that \( y \) is periodic and find its period \( q \).

ii. Suppose \( y \) has FS representation
\[
\forall t, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k \exp(k\omega_1 t),
\]
What is \( \omega_1 \)? Determine the \( Y_k \) in terms of the \( X_k \).

Answer to 7 (a) The formula is
\[
X_k = \frac{1}{p} \int_{0}^{p} x(t) e^{-ik\omega_0 t} dt. \tag{5}
\]

(b) We want \( y(t) = x(\alpha t) = y(t + q) = x(\alpha(t + q)) = x(t + p) \), so \( \alpha q = p \) or \( q = p/\alpha \). So the FS of \( y \) is
\[
y(t) = \sum_{k} Y_k e^{ik\omega_1 t}
= \sum_{k} X_k e^{ik\omega_0 t}
\]
from which \( \omega_1 = \alpha \omega_0 \) and \( Y_k = X_k \).

8. Give an example of a nonlinear, time-invariant system \( S \) that is not memoryless. Time is discrete.

(a) Show that \( S \) is nonlinear, time-invariant, and not memoryless.

(b) Suppose \( x : \text{Ints} \rightarrow \text{Reals} \) is periodic with period \( p \). Let \( y = S(x) \). Is \( y \) periodic?

(c) Suppose \( Q \) is another discrete-time, time-invariant system. Is the cascade composition \( S \circ Q \) time-invariant? Give a proof or a counterexample.

(d) Define the system \( R \) by reversing time: \( \forall x, n, R(x)(n) = S(x)(-n) \). Is \( R \) time-invariant? Why? If \( x \) is periodic as above and \( w = R(x) \), is \( w \) periodic? Why.
Answer to 8 One possible system is

\[ S(x)(n) = [x(n-1)]^2. \]

(a) \( S \) is clearly nonlinear since, if \( x(n-1) \neq 0 \), \( S(2x)(n) = 4[x(n-1)]^2 \neq 2[x(n-1)]^2 \).

\( S \) is time-invariant, since for any integer \( T \),

\[ S \circ D_T(x)(n) = [x(n-T-1)]^2 = D_T \circ S(x)(n). \]

\( S \) is not memoryless, because if it is there is \( f : \text{Reals} \rightarrow \text{Reals} \) with

\[ S(x)(n) = f(x(n)). \]

But this will not hold if we choose \( x, n, n-1 \) so that \( x(n) = 0 \) and \( [x(n-1)]^2 \neq f(0) \).

(b) Yes it is periodic, since

\[ S(x)(n+p) = D_{-p} \circ S(n) = S \circ D_{-p}(x)(n) = S(x)(n), \]

since \( D_{-p}x = x \) because \( x \) is periodic with period \( p \).

(c) The composition of any two time-invariant systems is periodic, since

\[ D_T \circ (Q \circ S) = Q \circ D_T \circ S = (Q \circ S) \circ D_T. \]

(d) \( R \) is not time-invariant, because

\[ D_T \circ R(x)(n) = R(x)(n-T) = S(x)(-n+T) = [x(-n+T-1)]^2 \]

\[ R \circ D_T(x)(n) = S \circ D_T(x)(-n) = [D_T(x)(-n-1)]^2 = [x(-n-1-T)]^2. \]

These two quantities are not equal for particular choices of \( x, n, T \).

\( w \) is periodic with the same period \( p \), because by part (b) \( S(x) \) is periodic with period \( p \), so

\[ w(n+p) = S(x)(-n-p) = S(x)(-n) = R(x)(n) = w(n). \]

9. You are given three kinds of building blocks for discrete-time systems: one-unit delay; gains; and adders.

(a) Use these building blocks to implement the system:

\[ y(n) = 0.5y(n-2) + x(n) + x(n-1). \]

(b) Take the outputs of the delay elements as the state and give a \([A, b, c']^T, d]\) representation of this system.

(c) You are allowed to set the output of the delay elements to any value at time \( n = 0 \). Select these values so that the output of your implementation is the solution \( y(n), n \geq 0 \) for any input \( x(n), n \geq 0 \) and initial conditions: \( y(-1) = 0.5, y(-2) = 0.8, x(-1) = 1 \). Now suppose \( x(0) = x(1) = x(2) = 0 \). Calculate \( y(0), y(1), y(2) \).
Figure 3: Implementation for problem 9

Answer to 9 (a) Figure 3 is one implementation.
(b) Taking $s(n) = [x(n-1) y(n-1) y(n-2)]^T$ and using (6) we get

$$
\begin{align*}
    s(n + 1) &= \begin{bmatrix} x(n) \\ y(n) \\ y(n-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix} s(n) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
    y(n) &= [1 \ 0 \ 0.5] s(n) + 1 \times x(n)
\end{align*}
$$

from which we can read off $A, b, c, d$.
(c) We take the initial state as $s(0) = [x(-1) y(-1) y(-2)]^T = [1 \ 0.5 \ 0.8]^T$. Then

$$
\begin{align*}
    y(0) &= e^T s(0) = [1 \ 0 \ 0.5] s(0) = 1.4 \\
    y(1) &= e^T A s(0) = 0.5^2 = 0.25 \\
    y(2) &= e^T A^2 s(0) = 0.7
\end{align*}
$$

One can also get these directly from (6).

10. An integrator can be used as a building block: For any input $x : \text{Reals} \rightarrow \text{Reals}$, its output is:

$$
\forall t \geq 0, \quad y(t) = y_0 + \int_0^t x(s) ds.
$$

The ‘initial condition’ $y(0)$ can be set.

Use integrators, gains and adders to implement the system:

$$
\frac{d^2 y}{dt^2}(t) - y(t) = x(t),
$$

with initial condition $y(0) = 1, \dot{y}(0) = 0.4$.

Hint First convert a differential equation into an integral equation and then implement.

Answer to 10 Figure 4 shows the implementation.
11. A periodic signal $x : \text{Reals} \rightarrow \text{Reals}$ is given by

$$x(t) = [1 + \cos(2\pi \times 10t)] \times \cos(2\pi \times 400t).$$

(a) What are the fundamental frequency $\omega_0$ and period $T_0$ of $x$? Calculate the Fourier Series of $x$ in the forms:

$$\forall t, \quad x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$$

$$= \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

Is $X_k = X^*_{-k}$?

(b) Suppose the LTI system $S$ has frequency response

$$\forall \omega, \quad H(\omega) = \begin{cases} 1, & \text{if } 2\pi \times 395 \leq |\omega| \leq 2\pi \times 405 \\ 0, & \text{otherwise} \end{cases}$$

Plot the magnitude and phase response of $H$. Repeat part 11a for $y$.

**Answer to 11** Using

$$\cos(x) \cos(y) = \frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y),$$

gives

$$x(t) = \cos(2\pi \cdot 400t) + \frac{1}{2} \cos(2\pi \cdot 390t) + \frac{1}{2} \cos(2\pi \cdot 410t),$$

from which

(a) $\omega_0 = 2\pi \cdot 10$ rad/sec and $t_0 = 0.1$ sec. Also

$$A_{39} = 0.5, \quad A_{40} = 0.5, \quad A_k = 0, \text{ else; } \forall k \phi_k = 0$$
and
\[ X_k = \frac{1}{2} A_k |e^{\phi_k sgn(k)} | \text{ in which } sgn(k) = 1, k \geq 0; = 0, k < 0. \]
So
\[ X_{39} = X_{-39} = X_{41} = X_{-41} = 0.25; \quad X_{40} = X_{-40} = 0.5; \quad X_k = 0, \text{ else.} \]

(b) This system is a bandpass filter, in which only sinusoids with frequencies within specified range go through unchanged and the others become 0. Thus
\[ \forall t, \quad y(t) = \cos(2 \pi \cdot 400 t); \quad \omega_0 = 2 \pi \cdot 400 \text{ rad/sec}; \quad T_0 = \frac{1}{400} \text{ sec}. \]
So,
\[ A_1 = 1; \quad A_k = 0, k \neq 1; \quad \phi_k = 0, \forall k; \]
\[ X_1 = X_{-1} = 0.5; \quad X_k = 0 \text{ else.} \]

12. Give the ABCD state space representation of a discrete-time system with frequency response \( H(\omega) \), where:
\[ H(\omega) = \frac{2 + e^{-j\omega}}{1 - 3e^{-3j\omega}} \]

**Hint:** First find a difference equation which has the given frequency response. Then find the state space representation.

**Answer to 12** From
\[ H(\omega)[1 - 3e^{-3j\omega}] = 2 + e^{-j\omega} \]
we see that \( H \) is the frequency response of the difference equation
\[ y(n) - 3y(n - 3) = 2x(n) + x(n - 1). \]
So we select
\[ s(n) = \begin{bmatrix} x(n-1) \\ y(n-1) \\ y(n-2) \\ y(n-3) \end{bmatrix} \]
\[ A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \]
\[ C^T = \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix} \quad D = 2 \]
13. You are given the signal $\forall t x(t) = \cos(20\pi t) + 1 - 2\sin(25\pi t)$ to use as input to a system with frequency response $H(\omega) = |\omega|$. Answer the following questions based on this setup.

(a) Indicate the Fourier series expansion (in cosine format) of $x$ by writing the nonzero values of $A_0$, $A_k$, and $\phi_k$ in the expansion $x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$.

(b) Indicate the Fourier series expansion (in complex exponential format) of $x(t)$ by writing the nonzero values of the complex coefficients $X_k$ in the expansion $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\omega_0 t}$.

(c) Give $y$, the output of the system with input $x$.

**Answer to 13**

(a) First rewrite $x(t) = \cos(20\pi t) + 1 - 2\sin(25\pi t)$ in terms of cosines:

$$x(t) = 1 + \cos(20\pi t) + 2 \cos(25\pi t + \frac{\pi}{2})$$

Next find the fundamental frequency. The largest frequency that evenly divides both $20\pi$ and $25\pi$ is $\omega_0 = 5\pi$. We rewrite $x(t)$ in terms of nonzero coefficients:

$$x(t) = 1 + \cos(4(5\pi)t + 0) + 2 \cos(5(5\pi)t + \frac{\pi}{2})$$

$$= A_0 + A_4 \cos(4\omega_0 t + \phi_4) + A_5 \cos(5\omega_0 t + \phi_5)$$

We see from above that $A_0 = 1$, $A_4 = 1$, $\phi_4 = 0$, $A_5 = 2$, $\phi_5 = \frac{\pi}{2}$, and all other $A_k$ and $\phi_k$ are zero.

(b) We can calculate the $X_k$’s directly, but since we’ve already calculated the $A_k$’s, let’s use them to derive the $X_k$’s. (See also page 302 in the text.) Note in particular that with complex exponentials, we have negative frequency and complex coefficients instead of phases, meaning that the $X_k$’s are complex and $k$ can be negative.

Recalling that

$$\cos(t) = \frac{e^{jt} + e^{-jt}}{2},$$

we can say that, for positive $k$:

$$A_k \cos(\omega_0 k t + \phi_k) = \frac{A_k e^{j\phi_k} e^{j\omega_0 k t}}{2} + \frac{A_k e^{-j\phi_k} e^{-j\omega_0 k t}}{2}$$

$$= X_k e^{j\omega_0 k t} + X_{-k} e^{j\omega_0 (-k) t}$$

In our case, we have three nonzero $A_k$. We start with $A_0$. Since $\cos(0) = e^{j0} = 1$, we conclude that $X_0 = A_0$.

For $A_4$, we relate the frequency components at $\omega = \pm 4\omega_0$:

$$1 \cos(4\omega_0 t) = \frac{1}{2} e^{4jt\omega_0 t} + \frac{1}{2} e^{-4jt\omega_0 t}$$

and conclude that $X_4 = 1/2$ and $X_{-4} = 1/2$. 

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And finally, for $A_5$ and $\phi_5$, we relate the frequency components at $\omega = \pm 5\omega_0$.

\[
2 \cos(5\omega_0 t) = e^{j\pi/2}e^{5j\omega_0 t} + e^{-j\pi/2}e^{-5j\omega_0 t} = ie^{5j\omega_0 t} - ie^{-5j\omega_0 t}
\]

and conclude that $X_5 = i$ and $X_5 = -i$.

(c) We can either apply the frequency response to the eigenfunctions or we can look at $x(t)$ directly and see how it behaves when sent through the system.

Let’s start with the latter approach.

Looking at $x(t) = \cos(20\pi t) + 1 - 2\sin(25\pi t)$, we see it has components at $\omega = 0$, $\omega = 20\pi$, and $\omega = 25\pi$. The frequency response is simple enough that we can see that the DC component (i.e. the component at $\omega = 0$) gets completely attenuated (i.e. multiplied by 0).

The other two components are scaled by the absolute value of their frequency, leading to:

\[
y(t) = (0)1 + (20\pi)\cos(20\pi t) - (25\pi)2\sin(25\pi t) = 20\pi \cos(20\pi t) - 50\pi \sin(25\pi t)
\]

If the frequency response had been more complicated, we may have preferred another approach:

We already have the complex exponential breakdown of the input signal, meaning that we know the input signal in terms of scaled eigenfunctions. We can therefore apply the frequency response:

\[
y(t) = H(0)X_0 + X_4H(4\omega_0)e^{4j\omega_0 t} + X_{-4}H(-4\omega_0)e^{-4j\omega_0 t} + X_5H(5\omega_0)e^{5j\omega_0 t} + X_{-5}H(-5\omega_0)e^{-5j\omega_0 t}
\]

\[
= 0 + \frac{1}{2}[20\pi|e^{20\pi t}| + \frac{1}{2} - 20\pi|e^{-20\pi t}| + |25\pi|e^{25\pi t} + | - 25\pi|(-i)e^{-25\pi t}]
\]

\[
= 20\pi e^{20\pi t} + e^{-20\pi t} + 50\pi(i^2)\frac{e^{25\pi t} - e^{-25\pi t}}{2i}
\]

\[
= 20\pi e^{20\pi t} + e^{-20\pi t} - 50\pi\frac{e^{25\pi t} - e^{-25\pi t}}{2i}
\]

\[
= 20\pi \cos(20\pi t) - 50\pi \sin(25\pi t)
\]

which is the same result as with the other method.

14. In the negative feedback system of figure 5 assume that $H(\omega) = [1 + i\omega]^{-1}$. Let $G$ be the closed-loop frequency response. For $K = 1, 10, 100$

(a) Plot the magnitude and phase response of $G$; and

(b) determine the bandwidth $\omega$ at which $\angle G(\omega) = \pi/4$. 

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**Answer to 14** The closed loop frequency response is

\[ G(\omega) = \frac{KH(\omega)}{1 + KH(\omega)} = \frac{K}{(K + 1) + i\omega}. \]

(a) So

\[ |G(\omega)| = \frac{K}{(K + 1)^2 + \omega^2}^{1/2}, \quad \angle G(\omega) = -\tan^{-1}\frac{\omega}{K + 1}. \]

(b) See figure 6

15. Determine the ‘gain’ \( k \) and the guard so that the output of the hybrid system is as shown in figure 7

**Answer to 15** The gain and guard are given in figure 7.
$s(0) = 0$

$\{s(t) \mid s(t) = 3\}$

$s(t) := 0$

$s(t) = 1$

$y(t) = 2s(t)$

$y(t)$

slope = 2

Figure 7: Hybrid system for problem 15