FORMULAS & TABLES

Discrete Fourier Series (DFS) Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) e^{-ik\omega_0 n} ,$$

where $p=\frac{2\pi}{\omega_0}$ and $\langle p\rangle$ denotes a suitable discrete interval of length p (i.e., an

interval containing p contiguous integers). For example, $\sum_{k=\langle p\rangle}$ may denote $\sum_{k=0}^{p-1}$

or
$$\sum_{k=1}^{p}$$
.

Continuous-Time Fourier Series (FS) Complex exponential Fourier series synthesis and analysis equations for a periodic continuous-time signal having period p:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt$$

where $p=\frac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable continuous interval of length p. For example, $\int_{\langle p \rangle} {\rm can\ denote}\ \int_0^p.$

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The DTFT and the Frequency Response of a DT LTI System Consider a real, discretetime LTI system having impulse response $h: \mathbb{Z} \to \mathbb{R}$. If the system has a frequency response $H: \mathbb{R} \to \mathbb{C}$, it is given by

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}, \quad \forall \omega \in \mathbb{R},$$

which is also known as the DTFT Analysis Equation. The impulse response of the system is given by

$$h(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} H(\omega) e^{i\omega n} d\omega ,$$

which is also known as the DTFT Synthesis Equation.

The CTFT and the Frequency Response of a CT LTI System Consider a real, continuoustime LTI system having impulse response $h : \mathbb{R} \to \mathbb{R}$. If the system has a frequency response $H : \mathbb{R} \to \mathbb{C}$, it is given by

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt, \quad \forall \omega \in \mathbb{R},$$

which is also known as the CTFT Analysis Equation. The impulse response of the system is given by

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega,$$

which is also known as the CTFT Synthesis Equation.

Duality Let the signal $x : \mathbb{R} \to \mathbb{C}$ have CTFT $X : \mathbb{R} \to \mathbb{C}$. More compactly, let

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega),$$

where \mathcal{F} denotes the Fourier transform. Then

$$X(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega).$$

Parseval-Plancherel-Rayleigh Identity Let the signals $x,y:\mathbb{R}\to\mathbb{C}$ have respective CTFTs $X,Y:\mathbb{R}\to\mathbb{C}$. Then

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) Y^*(\omega) d\omega.$$

Time domain	Frequency domain
$\forall n \in \mathbb{Z}, x(n) \text{ is real}$	$\forall \ \omega \in \mathbb{R}, X(\omega) = X^*(-\omega)$
$\forall n \in \mathbb{Z}, x(n) = x^*(-n)$	$orall \ \omega \in \mathbb{R}, X(\omega) ext{ is real}$
$\forall n \in \mathbb{Z}, y(n) = x(n-N)$	$\forall \omega \in \mathbb{R}, Y(\omega) = e^{-i\omega N} X(\omega)$
$\forall n \in \mathbb{Z}, y(n) = e^{i\omega_1 n} x(n)$	$\forall \omega \in \mathbb{R}, Y(\omega) = X(\omega - \omega_1)$
$\forall n \in \mathbb{Z},$ $y(n) = \cos(\omega_1 n) x(n)$	$\forall \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) + X(\omega + \omega_1))/2$
$\forall n \in \mathbb{Z},$ $y(n) = \sin(\omega_1 n) x(n)$	$Y(\omega) = (X(\omega - \omega_1) - X(\omega + \omega_1))/2i$
$\forall n \in \mathbb{Z},$ $x(n) = ax_1(n) + bx_2(n)$	$\forall \omega \in \mathbb{R},$ $X(\omega) = aX_1(\omega) + bX_2(\omega)$
$\forall n \in \mathbb{Z}, y(n) = (h * x)(n)$	$\forall \omega \in \mathbb{R}, Y(\omega) = H(\omega)X(\omega)$
$\forall n \in \mathbb{Z}, y(n) = x(n)p(n)$	$Y(\omega) = \frac{1}{2\pi} \int_{0}^{2\pi} X(\Omega) P(\omega - \Omega) d\Omega$
$\forall n \in \mathbb{Z}, \\ y(n) = \\ \left\{ \begin{array}{ll} x(n/N) & n \text{ is a multiple of } N \\ 0 & \text{otherwise} \end{array} \right.$	$\forall \ \omega \in \mathbb{Z},$ $Y(\omega) = X(N\omega)$

Table 1: Properties of the DTFT.

Signal	DTFT
$\forall n \in \mathbb{Z}, x(n) = \delta(n)$	$\forall \omega \in \mathbb{R}, X(\omega) = 1$
$\forall n \in \mathbb{Z}, \\ x(n) = \delta(n - N)$	$\forall \omega \in \mathbb{R}, X(\omega) = e^{-i\omega N}$
$\forall n \in \mathbb{Z}, x(n) = 1$	$\forall \omega \in \mathbb{R},$ $X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$
$\forall n \in \mathbb{Z},$ $x(n) = a^n u(n), a < 1$	$\forall \omega \in \mathbb{R},$ $X(\omega) = \frac{1}{1 - ae^{-i\omega}}$
$\forall \ n \in \mathbb{Z},$ $x(n) = \left\{ \begin{array}{ll} 1 & \text{if } n \leq M \\ 0 & \text{otherwise} \end{array} \right.$	$X(\omega) = \frac{\sin(\omega(M+0.5))}{\sin(\omega/2)}$
$\forall n \in \mathbb{Z},$ $x(n) = \frac{\sin(Wn)}{\pi n}, 0 < W < \pi$	$orall \ \omega \in [-\pi,\pi],$ $X(\omega) = \left\{egin{array}{ll} 1 & ext{if } \omega \leq W \ 0 & ext{otherwise} \end{array} ight.$

Table 2: Discrete time Fourier transforms of key signals. The function \boldsymbol{u} is the unit step.

Time domain	Frequency domain
$\forall \ t \in \mathbb{R}, x(t) \ ext{is real}$	$\forall \omega \in \mathbb{R}, X(\omega) = X^*(-\omega)$
$\forall t \in \mathbb{R}, x(t) = x^*(-t)$	$orall \ \omega \in \mathbb{R}, X(\omega) ext{ is real}$
$\forall t \in \mathbb{R}, y(t) = x(t - T)$	$\forall \omega \in \mathbb{R}, Y(\omega) = e^{-i\omega T} X(\omega)$
$\forall t \in \mathbb{R}, y(t) = e^{i\omega_1 t} x(t)$	$\forall \ \omega \in \mathbb{R}, Y(\omega) = X(\omega - \omega_1)$
$\forall t \in \mathbb{R}, \\ y(t) = \cos(\omega_1 t) x(t)$	$\forall \omega \in \mathbb{R},$ $Y(\omega) = (X(\omega - \omega_1) + X(\omega + \omega_1))/2$
$\forall t \in \mathbb{R}, \\ y(t) = \sin(\omega_1 t) x(t)$	$Y(\omega) = (X(\omega - \omega_1) - X(\omega + \omega_1))/2i$
$\forall t \in \mathbb{R},$ $x(t) = ax_1(t) + bx_2(t)$	$\forall \omega \in \mathbb{R},$ $X(\omega) = aX_1(\omega) + bX_2(\omega)$
$\forall t \in \mathbb{R}, y(t) = (h * x)(t)$	$\forall \omega \in \mathbb{R}, Y(\omega) = H(\omega)X(\omega)$
$\forall t \in \mathbb{R}, y(t) = x(t)p(t)$	$Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) P(\omega - \Omega) d\Omega$
$\forall t \in \mathbb{R}, \\ y(t) = x(at)$	$Y(\omega) = \frac{1}{ a } X(\omega/a)$

Table 3: Properties of the CTFT.

Signal	CTFT
$\forall t \in \mathbb{R}, x(t) = \delta(t)$	$\forall \omega \in \mathbb{R}, X(\omega) = 1$
$\forall t \in \mathbb{R}, x(t) = \delta(t - \tau), \ \tau \in \mathbb{R}$	$\forall \omega \in \mathbb{R}, X(\omega) = e^{-i\omega\tau}$
$\forall t \in \mathbb{R}, x(t) = 1$	$\forall \ \omega \in \mathbb{R}, X(\omega) = 2\pi \delta(\omega)$
$\forall t \in \mathbb{R},$ $x(t) = e^{-at} u(t), a > 0$	$\forall \omega \in \mathbb{R},$ $X(\omega) = \frac{1}{a + i\omega}$
$\forall t \in \mathbb{R},$ $x(t) = e^{-a t }, a > 0$	$\forall \omega \in \mathbb{R},$ $X(\omega) = \frac{2a}{a^2 + \omega^2}$
$orall t \in \mathbb{R},$ $x(t) = \operatorname{sgn}(t), orall t$	$orall \ \omega \in \mathbb{R},$ $X(\omega) = rac{2}{i\omega}$
$\forall t \in \mathbb{R},$ $x(t) = \begin{cases} \pi/a & \text{if } t \le a \\ 0 & \text{otherwise} \end{cases}$	$\forall \omega \in \mathbb{R},$ $X(\omega) = \frac{2\pi \sin(a\omega)}{a\omega}$
$\forall t \in \mathbb{R},$ $x(t) = \frac{\sin(\pi t/T)}{\pi t/T},$	$\forall \omega \in \mathbb{R},$ $X(\omega) = \left\{ egin{array}{ll} T & ext{if } \omega \leq \pi/T \\ 0 & ext{otherwise} \end{array} \right.$

Table 4: Continuous time Fourier transforms of key signals. The function \boldsymbol{u} is the unit step.