Sample problems for Fall 99 MT2.

The problems below cover Chapters 8,9,10.

Chapter 8. Fourier Transform

The problems in this section require direct application of definitions, evaluation of integrals or series, and plotting.

1. **E** Consider the continuous-time periodic function x, where for $n = 0, \pm 3, \pm 6. \pm 9, \cdots$,

$$x(t) = \begin{cases} 1, & n \le t < n+1 \\ 0, & n+1 \le t < n+2 \\ -1, & n+2 \le t < n+3 \end{cases}$$

   (a) Sketch x. What is its period in seconds? What is its fundamental frequency in radians/sec and in Hz?

   (b) x has an exponential Fourier series of the form

$$\forall t \in Reals, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}. \tag{1}$$

   What is ω_0?

   (c) Write down a formula for the coefficients X_k.

   (d) How would you quickly find X_{-k} from X_k?

   (e) Find the Fourier series coefficients in (1).

   (f) The signal y obtained by delaying x by 1 second,

$$\forall t, \quad y(t) = x(t-1),$$

   is periodic and has the Fourier series representation

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_0 t}.$$

   Find the Y_k in terms of the X_k obtained above.

   (Hint. See (8.1), (8.2).)

2. **E** Consider the discrete-time periodic signal x, where for $m = 0, \pm 3, \pm 6, \cdots$,

$$x(n) = \begin{cases} 1, & n = m \\ 0, & n = m+1 \\ -1, & n = m+2 \end{cases}$$

   (a) Sketch x. What is its period p in samples? What is its fundamental frequency in radians/sample?

(b) x has a DTFT expansion

$$\forall n \in Ints, \quad x(n) = \sum_{k=0}^{p-1} X_k e^{jk\omega_0 n}. \tag{2}$$

What is ω_0?

(c) Write down the formula for the coefficients X_k.

(d) Find the coefficients X_k in (2).

(e) The signal y obtained by delaying x by 1 sample,

$$\forall n, \quad y(n) = x(n-1),$$

is periodic and has the DTFT representation

$$y(n) = \sum_{k=0}^{p-1} Y_k e^{jk\omega_0 n}.$$

Find the Y_k in terms of the X_k obtained above.

(Hint. See (8.5), (8.6).)

3. **E** Consider the continuous-time signal $x : Reals \to Reals$,

$$\forall t, \quad x(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where α is a fixed positive number.

(a) Sketch x and mark $x(0)$.

(b) Write down the formula for the Fourier Transform $X(\omega), \omega \in Reals$, of x. What are the units of ω?

(c) Calculate the Fourier Transform.

(d) Plot the amplitude spectrum $|X(\omega)|$ and the phase spectrum $\angle X(\omega)$, for $\omega \in Reals$.

(e) Consider the signal y obtained by advancing x by 1 second,

$$\forall t, \quad y(t) = x(t+1),$$

and let its Fourier Transform be $Y(\omega)$. Write Y in terms of X. Show that for all ω, $|Y(\omega)| = |X(\omega)|$.

(f) Suppose the signal z is given by time-reversal

$$\forall t, \quad z(t) = x(-t).$$

Obtain its Fourier Transform Z in terms of X.

(g) Suppose the signal w is given by frequency shifting,

$$\forall t, \quad w(t) = x(t) \times \cos(2\pi 60 t).$$

Obtain its Fourier Transform W in terms of X.

(h) Plot $|W(\omega)|$. Mark $\pm 2\pi \times 60$ rad/sec on your plot.

(Hint. See (8.11), (8.12), and FT properties on p. 156.)

4. **E** The signal $x \in DiscSignal$ is given by

$$\forall n \in Ints, \quad x(n) = 2^{-|n|}.$$

(a) Plot x. Mark $x(0)$.

(b) x has the DTFT $X(\omega)$, $\omega \in Reals$. What is the unit of ω?. What is the formula for X?

(c) Calculate the DTFT X. (Hint. Separate the infinite sum in the DTFT in two parts, $n \le 0, n < 0$, and use the geometric sum identity: $\sum_{k=0}^{\infty} \beta^k = 1/(1-\beta)$.)

(d) Plot the magnitude spectrum $\omega \mapsto |X(\omega)|$ and the phase spectrum $\omega \mapsto \angle X(\omega)$.

Chapter 8, contd.

The problems in this section require familiarity with the properties of Fourier Transforms (pp. 156-7).

1. **T** Let $sinc_T$ denote the function,

$$\forall t, \quad sinc_T(t) = \frac{\sin(\pi t/T)}{(\pi t/T)}.$$

We know that the Fourier Tranform of $sinc_T$ is the function $Gate_{\pi/T}$:

$$\forall \omega, \quad Gate_{\pi/T}(\omega) = \begin{cases} T, & \text{if } |\omega| \le \pi/T \\ 0, & \text{if } |\omega| > \pi/T \end{cases}$$

(a) Plot $sinc_T$ and $Gate_{\pi/T}$ and carefully mark some interesting values on your plot.

(b) Use the fact that

$$x * y \leftrightarrow XY$$

to find the Fourier Transform of $sinc_{T_1} * sinc_{T_2}$ with $T_1 \le T_2$.

2. Recall that if x is the input and y is the response of an LTI system with frequency response $H(\omega)$, then $Y = H \times X$, where X and Y are the Fourier Transforms of x and y. Perform the following operations:
Step 1. Obtain z by time-reversing y, i.e. $z(t) = y(-t)$.
Step 2. Now input z to the same LTI system and obtain the response w.
Step 3. Time-reverse w and obtain v, i.e $v(t) = w(-t)$.

(a) Obtain the Fourier Transforms of z, w, and v in terms of X and H. (Hint. Use Time-Scale property (with $\alpha = -1$) on p. 156 and recall the real signals property on p. 157.)

(b) Show that $V(\omega) = |H(\omega)|^2 X(\omega)$, for all ω.

(c) In particular, show that if $|H(\omega)|^2 = 1$, for all ω, then $v(t) = x(t)$, for all t.

3. Suppose $x \in DiscSignals$ has DTFT $X(\omega), \omega \in Reals$. Suppose you know that

$$X(\omega) = \begin{cases} 1 - \omega/(\pi/2), & 0 \le \omega \le \pi/2 \\ 0, & \pi/2 \le \omega \le \pi. \end{cases}$$

(a) Use the fact that $X(-\omega) = X(\omega)^*$ and that X is periodic with period 2π rads/sec, to plot $X(\omega)$ for $-4\pi \le \omega \le 4\pi$ rads/sec.

(b) Suppose the discrete signal y is obtained by up-sampling x (a zero-valued sample is inserted between successive samples of x):

$$\forall n, \quad y(n) = \begin{cases} x(n/2), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Express the DTFT Y of y in terms of X.

(c) Plot $Y(\omega)$ for $-4\pi \le \omega \le 4\pi$.

4. Suppose $z \in DiscSignals$ is obtained from $x \in DiscSignals$ by retaining every even-numbered sample of x and replacing every odd-numbered sample value of x by 0, i.e.

$$\forall n, \quad z(n) = \begin{cases} x(n), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Obtain the DTFT Z of z in terms of X, the DTFT of x. Hint. Follow these steps:
Step 1. Check that z is given by

$$\forall n, \quad z(n) = \frac{1}{2} x(n)[1 + e^{i\pi n}].$$

Step 2. Use this relation to show that

$$\forall \omega, \quad Z(\omega) = \frac{1}{2}[X(\omega) + X(\omega - \pi)].$$

5. Plot $Z(\omega)$ for $-4\pi \le \omega \le 4\pi$.

6. Suppose $z \in DiscSignals$ is such that $z(n) = 0$ whenever n is odd. Suppose w is obtained from z by deleting z's odd-numbered, zero-valued samples, i.e.

$$\forall n, \quad w(n) = z(2n).$$

Obtain the DTFT W of w in terms of Z.

7. Obtain w from x by deleting every odd-numbered sample of x,

$$\forall n, \quad w(n) = x(2n).$$

We say that w is obtained by down-sampling x. Express W in terms of X. (Hint. Use the previous two problems.)

Chapter 9. Sampling and aliasing.

The problems in this section require familiarity with the calculations that lead to Figures 9.2, 9.3, 9.6-9.8.

1. The signal $\forall t, x(t) = \cos(2\pi \times 10t)$, is sampled every T seconds, and the resulting discrete signal is y.

   (a) What is the domain of y? Write an expression for y.

   (b) Suppose $T = 10^{-3}$. Give a rough sketch of y.

   (c) Suppose y is input to $IdealInterpolator_T$ and the resulting continuous-time signal is z. Write an expression for z.

   (d) How small should be the sampling period T to guarantee that $z = x$.

   (e) Suppose $T = 1/3$ sec, what is z? (Hint. With $T = 1/3$ the sampling frequency is less that twice the Nyquist frequency of 10 Hz, so there will be aliasing as determined by Figure 9.2 or 9.3.)

2. Consider the signal

$$\forall t, \quad x(t) = \cos(2\pi \times 10t) + \cos(2\pi \times 12t) + \cos(2\pi \times 14t).$$

   The signal is sampled every T seconds and the resulting discrete signal y is input to $IdealInterpolator_T$ and the continuous-time signal z is obtained.

   (a) Suppose $T = 10^{-3}$. What is z?

   (b) Suppose $T = 1/26$ sec. What is z?

   (c) Suppose $T = 1/22$ sec. What is z?

   (d) Suppose $T = 1/18$ sec. What is z?

   (Hint. The signal x is the sum of three sinusoids. The signal z is therefore the sum of the responses to these three sinusoids. Each response can be obtained following the preceding problem.)

3. A signal x is bandlimited to frequency ω_0 radians/second, i.e. $X(\omega) = 0$ for $|\omega| > \omega_0$, as illustrated in Figure 1. The signal x is periodically sampled with frequency ω_s radians/sec and the resulting signal is called $y \in DiscreteSignals$.

   (a) What is the sampling period T_s in seconds? Write an expression for y.

   (b) Express the DTFT Y of y in terms of X. (Hint. See (9.5).)

   (c) Suppose $\omega_s = 2\omega_0 + 100$. Sketch the magnitude spectrum $|Y(\omega)|$ and carefully mark $\omega_0, \omega_s/2, \omega_s$ on your plot.

   (d) Suppose again that $\omega_s = 2\omega_0 + 100$. Suppose y is passed through $IdealInterpolator_{T_s}$ and the resulting continuous-time signal is z. What is the relation between z and

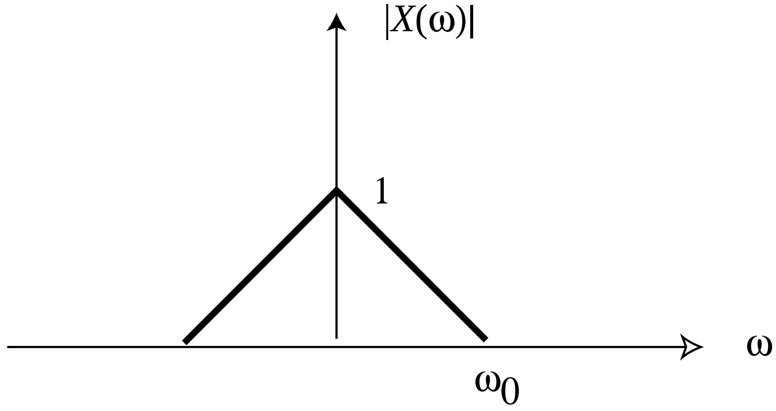

Figure 1: Magnitude spectrum of a signal bandlimited to ω_0.

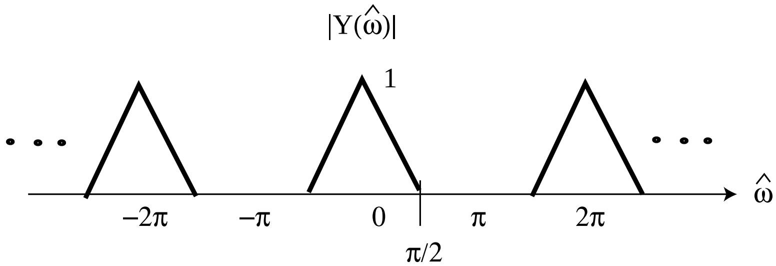

Figure 2: Plot of $|Y|$.

x? Hint. See Figure 9.6 or the expression for Z on p. 171. Note the typo there: The correct expression is:

$$
\begin{aligned}
Z(\omega) &= \begin{cases} TY(\omega T) & -\pi/T < \omega < \pi/T \\ 0 & otherwise \end{cases} \\
&= \begin{cases} \sum_{k=-\infty}^{\infty} X(\omega - 2\pi k/T) & -\pi/T < \omega < \pi/T \\ 0 & otherwise \end{cases}
\end{aligned}
$$

(e) Now suppose $\omega_s = 2\omega_0 - 100$. Suppose y is passed through $IdealInterpolator_{T_s}$ and the resulting signal is z. Give an expression for Z in terms of X (Hint. Same as above.) Now sketch $|Z(\omega)|$ and carefully mark $\omega_0, \omega_s/2, \omega$ on your plot.

Chapter 9 contd.

These problems require a deeper understanding of sampling.

1. Let $y : Ints \to Reals$ be any signal, with DTFT $Y(\hat{\omega})$, where $\hat{\omega}$ is in radians/sample (as on p.170). Then Y is periodic. Suppose $|Y|$ is as shown in Figure 2. Suppose that y is passed through $IdealInterpolator_T$ and let z be the continuous-time output signal.

(a) Suppose $T = 10^{-3}$ sec. What is the relationship between $z(nT)$ and $y(n)$?

(b) Let $Z(\omega)$ be the Fourier Transform of z. The units of ω is radians/sec. Is z bandlimited? Plot $|Z(\omega)|$ knowing $|Y(\hat{\omega})|$ from Figure 2. Carefully mark the following frequencies on your plot: $\omega = \pi/(2T), \pi/T, 2\pi/T$.

(c) Repeat the previous parts for $T = 1$ sec and $T = 10^{-6}$ sec.

Hint. To answer this problem you must realize that no matter what T is and what discrete signal y is input to $IdealInterolator_T$, the result is a continuous-time signal z with two properties: (1) it interpolates y, i.e. $\forall n, z(nT) = y(n)$, and (2) z is band-limited to π/T radians/sec, i.e. $Z(\omega) = 0$, for $|\omega| > \pi/T$.

2. Suppose your voice is bandlimited to 15 KHz. Suppose you record a song with a sampling frequency of 30 KHz.

   (a) Now suppose you play it back through $IdealInterpolator_T$ where $T = 1/30,000$ sec. How will it sound compared to your original singing?

   (b) Now suppose you play it back through $IdealInterpolator_T$ with $T = 1/40,000$ sec. How will it sound now? Why?

Chapter 10

(a) Construct a state machine with $Inputs = \{0, 1\}$, $Outputs = \{yes, no, absent\}$ such that the machine outputs *yes* if the three most recent inputs are 000, outputs *no* if the three most recent inputs are 111, and outputs *absent* otherwise. In other words, if the input is

$$u(0), u(1), u(2), \cdots$$

then the output after the nth step is

$$y(n) = \begin{cases} yes, & \text{if } u(n-2)u(n-1)u(n) = 000 \\ no, & \text{if } u(n-2)u(n-1)u(n) = 111 \\ absent, & \text{otherwise} \end{cases}$$

What is the output sequence of your machine when the input sequence is

$$0011000111 \cdots ?$$

(b) For the machine you've constructed in the previous problem what are its *States* and express the *nextState* and *nextOutput* functions as tables.