8.4  (a) No, since the response to an impulse includes non-zero samples earlier than time zero.
(b) The frequency response is the DTFT of the impulse response,
\[ H(\omega) = \sum_{m=-\infty}^{\infty} h(m)e^{-im\omega} \]
\[ = \sum_{m=-\infty}^{\infty} (\delta(m-1)/2 + \delta(m+1)/2)e^{-im\omega} \]
\[ = (e^{-i\omega} + e^{i\omega})/2 \]
\[ = \cos(\omega). \]

This is periodic with period \(2\pi\) because
\[ \forall \omega \in \text{Reals}, \quad \cos(\omega + 2\pi) = \cos(\omega). \]

(c) The fundamental frequency \(\omega_0 = \pi/2\), in units of radians per sample. To get the Fourier series coefficients, just write the signal as a sum of complex exponentials,
\[ x(n) = (1/2)e^{-i\pi n} + (i/2)e^{-i\pi n/2} + 2 - (i/2)e^{i\pi n/2} + (1/2)e^{-i\pi n}, \]
from which we can read off the coefficients,
\[ X_{-2} = 1/2 \]
\[ X_{-1} = i/2 \]
\[ X_0 = 2 \]
\[ X_1 = -i/2 \]
\[ X_2 = 1/2. \]

The rest of the coefficients are zero.

(d) The Fourier series coefficients of the output will be the above Fourier series coefficients multiplied by \(H(\omega)\) for the corresponding value of \(\omega\). This yields
\[ y(n) = -(1/2)e^{-i\pi n} + 2 - (1/2)e^{i\pi n} \]
\[ = 2 - \cos(\pi n). \]

8.5 We can calculate the CTFT of the impulse response,
\[ H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt \]
\[ = \int_{0}^{3} (1/3)e^{-i\omega t} dt \]
\[ = (1 - e^{-i3\omega})/(3i\omega). \]

The following Matlab code plots the magnitude response:
Figure 1: Magnitude response of a 3-second continuous-time moving average.

\[ f = [-5:1/100:5]; \]
\[ H = (1 - \exp(-i*3*2*pi*f)) / (3*i*2*pi*f); \]
\[ \text{plot}(f, \text{abs}(H)); \]

Note that this gives a “Warning: Divide by zero” at frequency 0, but generates a correct plot anyway. You can use L’Hopital’s rule to find that the value at frequency zero is 1. The plot is shown in figure 1.

8.6 (a) Using convolution,

\[ y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \]
\[ = \int_{-\infty}^{\infty} (\delta(\tau - 1) + \delta(\tau - 2))x(t - \tau)d\tau \]
\[ = \int_{-\infty}^{\infty} \delta(\tau - 1)x(t - \tau)d\tau + \int_{-\infty}^{\infty} \delta(\tau - 2)x(t - \tau)d\tau \]
\[ = x(t - 1) + x(t - 2), \]

using the sifting rule.
Figure 2: The magnitude frequency response of an LTI system with impulse response \( h(t) = \delta(t - 1) + \delta(t - 2) \).

(b) The frequency response is the CTFT of the impulse response,

\[
H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt = \int_{-\infty}^{\infty} (\delta(t - 1) + \delta(t - 2))e^{-i\omega t} dt = e^{-i\omega} + e^{-i2\omega},
\]

using the sifting rule.

(c) The following Matlab code creates the plot:

```matlab
f = [-5:1/100:5];
H = (exp(-i*2*pi*f)+exp(-i*2*2*pi*f));
plot(f,abs(H));
```

which yields the plot shown in figure 2.

9.8 (a) Note that

\[
X(-\omega) = i \sin(-K\omega) = -i \sin(K\omega) = X^*(-\omega),
\]
using the fact that $\sin(\theta) = -\sin(-\theta)$. Thus, $X$ is conjugate symmetric, which implies that $x$ is real.

(b) Using Euler’s relation,

$$X(\omega) = (e^{iK\omega} - e^{-iK\omega})/2.$$ 

We can recognize the inverse DTFT of each of these terms to get

$$x(n) = (\delta(n + K) - \delta(n - K))/2$$

where $\delta$ is the Kronecker delta function.

9.9 First, note that $y$ is periodic with period $p$, just as $x$ is. Its Fourier series coefficients are given by the formula

$$Y_m = \frac{1}{p} \int_{0}^{p} y(t)e^{-im\omega_0 t} dt$$

$$= \frac{1}{p} \int_{0}^{p} x(t - \tau)e^{-im\omega_0 t} dt$$

$$= \frac{1}{p} \int_{-\tau}^{p-\tau} x(t)e^{-im\omega_0 (t+\tau)} dt$$

$$= e^{-im\omega_0 \tau} \frac{1}{p} \int_{-\tau}^{0} x(t)e^{-im\omega_0 t} dt + e^{-im\omega_0 \tau} \frac{1}{p} \int_{0}^{p} x(t)e^{-im\omega_0 t} dt$$

$$= e^{-im\omega_0 \tau} X_m,$$

where we have changed variables in the integral (replacing $t$ with $t - \tau$), and then changed the limits from $-\tau$ to $p - \tau$ to $0$ to $p$. The change of limits is valid because we are integrating over one cycle of a periodic function, so it does not matter where the integral begins. The end result is

$$Y_m = e^{-im\omega_0 \tau} X_m,$$

so just as with a CTFT, a time delay affects Fourier series coefficients by multiplying them by a complex exponential.

9.10 Use the inverse CTFT,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{i\omega t} d\omega$$
\[
= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} e^{i\omega t} d\omega
\]
\[
= \frac{T}{2\pi iT} [e^{i\pi T} - e^{-i\pi T}]
\]
\[
= \frac{\sin(t\pi/T)}{t\pi/T}.
\]

9.11 Use the CTFT,

\[
Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-i\omega t} dt
\]
\[
= \int_{-\infty}^{\infty} X(t)e^{-i\omega t} dt
\]

so

\[
\frac{1}{2\pi} Y(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t)e^{i\omega t} dt
\]
\[
= x(\omega),
\]
recognizing this as an inverse CTFT with symbols \(\omega\) and \(t\) swapped. Thus,

\[
\frac{1}{2\pi} Y(-\omega) = x(\omega)
\]

which implies that

\[
Y(\omega) = 2\pi x(-\omega).
\]

9.12 Define

\[
y(t) = X(t) = 2\pi \frac{\sin(\alpha t)}{\alpha t}.
\]

From exercise , with \(\pi/T\) replaced by \(\alpha\),

\[
Y(\omega) = \begin{cases} 
(2\pi)\pi/\alpha, & \text{if } |\omega| \leq \alpha \\
0, & \text{if } |\omega| > \alpha 
\end{cases}
\]

From exercise ,

\[
Y(\omega) = 2\pi x(-\omega)
\]
so

\[ x(t) = \frac{1}{2\pi} Y(-t). \]

Hence,

\[ x(t) = \begin{cases} 
\pi/a, & \text{if } |t| \leq a \\
0, & \text{if } |t| > a 
\end{cases} \]

10.2 Note that \( \cos(\theta) = \cos(-\theta) \). Therefore,

\[ \cos(-2\pi 440nT + \phi) = \cos(2\pi 440nT - \phi). \]

Thus, \( f = 440 \) and \( \theta = -\phi. \)

10.6 (a) The sketch is shown below:

\[ H(2\pi f) \]

The height of each of the peaks is \( 1/T \), which in this case is 40,000.

(b) The sketch is shown below:

\[ H(2\pi f) \]

The height of each of the peaks is \( 1/T \), which in this case is 20,000.

(c) The sketch is shown below:

\[ H(2\pi f) \]

The height of each of the peaks is \( 1/T \), which in this case is 15,000. Notice that the overlapping CTFTs caused aliasing distortion.

1. (a) The impulse response is shown below:

\[ h(n) \]
(b) Use convolution to relate the input and output
\[
y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)
\]
\[
= x(n) + 2x(n-1),
\]
using the sifting rule. When the input is the unit step, this becomes
\[
y(n) = u(n) + 2u(n-1) = \begin{cases} 
0 & \text{if } n < 0 \\
1 & \text{if } n = 0 \\
3 & \text{if } n \geq 1 
\end{cases}
\]
Here is a plot:

(c) If the input is \(r\), then the output is
\[
y(n) = r(n) + 2r(n-1) = \begin{cases} 
0 & \text{if } n \leq 0 \\
3n - 2 & \text{if } n \geq 1 
\end{cases}
\]
Here is a plot:

(d) The frequency response is the DTFT of the impulse response,
\[
H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k}
\]
\[
= 1 + 2e^{-i\omega}.
\]
(e) For all \(\omega \in \text{Reals},\)
\[
H(\omega + 2\pi) = 1 + 2e^{-i(\omega + 2\pi)}
\]
\[
= 1 + 2e^{-i\omega}e^{-i2\pi}
\]
\[
= 1 + 2e^{-i\omega}, \text{ since } e^{-i2\pi} = 1
\]
\[
= H(\omega).
\]
(f)
\[ H(-\omega) = 1 + 2e^{i\omega} \]
\[ = (1 + 2e^{-i\omega})^* \]
\[ = H^*(\omega). \]

(g) The magnitude response is
\[ |H(\omega)| = |1 + 2e^{-i\omega}| \]
\[ = |1 + 2\cos(\omega) - 2i\sin(\omega)| \]
\[ = \sqrt{(1 + 2\cos(\omega))^2 + (2\sin(\omega))^2} \]
\[ = \sqrt{1 + 4\cos(\omega) + 4\cos^2(\omega) + 4\sin^2(\omega)} \]
\[ = \sqrt{5 + 4\cos(\omega)}. \]

We have used the facts that for real numbers \( a \) and \( b \),
\[ |a + ib| = \sqrt{a^2 + b^2} \]
and for any \( \omega \in \text{Reals} \),
\[ \cos^2(\omega) + \sin^2(\omega) = 1. \]

(h) The phase response is
\[ \angle H(\omega) = \angle(1 + 2e^{-i\omega}) \]
\[ = \angle(1 + 2\cos(\omega) - 2i\sin(\omega)) \]
\[ = \tan^{-1}(-2\sin(\omega)/(1 + 2\cos(\omega))) \]
\[ = -\tan^{-1}(2\sin(\omega)/(1 + 2\cos(\omega))). \]

We have used the fact that for real numbers \( a \) and \( b \),
\[ \angle(a + ib) = \tan^{-1}(b/a). \]

(i) The output will be
\[ y(n) = |H(\pi/2)|\cos(\pi n/2 + \pi/6 + \angle H(\pi/2)) + |H(\pi)|\sin(\pi n + \pi/3 + \angle H(\pi)). \]

In this case,
\[ H(\pi/2) = 1 - 2i \]
and
\[ H(\pi) = -1. \]

So
\[ |H(\pi/2)| = \sqrt{5}, \quad \angle H(\pi/2) = -\tan^{-1}(2) \approx 1.107 \]
and
\[ |H(\pi)| = 1, \quad \angle H(\pi) = \pi. \]

Hence,
\[ y(n) = \sqrt{5}\cos(\pi n/2 + \pi/6 + 1.107) + \sin(\pi n + \pi/3 + \pi). \]