EECS 20. Midterm No. 2 Solution  
April 11, 2003.

1. 15 points For the following hybrid system sketch

(a) 10 points the state trajectory (both the mode and the continuous state) and
(b) 5 points the output signal for $0 \leq t \leq 3$.

2. 15 points, 5 points each part
   Give the units of period and frequency below

(a) Consider the discrete-time signal $x$ given by
   \[
   \forall n \in \text{Integers}, \quad x(n) = \cos(\omega n).
   \]
   For what values of $\omega$ is $x$ periodic, and what is the period?
   $x$ is periodic with period $p \text{ samples}$ if $\omega p$ is a multiple of $2\pi$, i.e. if $\omega = 2m\pi/p \text{ rad/sample}$ for integers $m, p$. To get the smallest period $p, m, p$ must be coprime.
(b) Consider the discrete-time signal \( x \) given by
\[
\forall n \in \text{Integers}, \quad x(n) = 1 + \cos(4\pi n/9).
\]
What is its period \( p \) and what is its fundamental frequency?
The period is \( p = 9 \) samples, and the fundamental frequency is \( \omega_0 = 2\pi/9 \) rads/sample.
The signal has the Fourier series representation
\[
\forall n, \quad y(n) = A_0 + \sum_{k=1}^{[p/2]} A_k \cos(k\omega_0 n + \phi_k).
\]
Identify \( \omega_0, A_0, A_k, \phi_k \).
\( A_0 = 1, A_2 = 1, \phi_2 = 0, \) other coefficients are 0.

(c) Consider the continuous-time periodic signal \( y \) given by
\[
\forall t \in \text{Reals}, \quad y(t) = \cos 5t + \sin 3t.
\]
What is its period and what is its fundamental frequency?
Its fundamental frequency is \( \omega_0 = \gcd\{3, 5\} = 1 \) rad/sec and the period is \( p = 2\pi/p = 2\pi \) sec.
The Fourier series representation of \( y \) is
\[
\forall t, \quad y(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k).
\]
Identify \( \omega_0, A_0, A_k, \phi_k \).
We have
\[
\forall t, \quad y(t) = \cos 5t + \cos(3t - \pi/2).
\]
So,
\( A_3 = 1, \phi_3 = -\pi/2, A_5 = 1, \phi_5 = 0, \) other coefficients are 0.

3. 15 points, 3 points each part Consider the following discrete-time systems with input \( x : \) Integers \( \rightarrow \) Reals and output \( y : \) Integers \( \rightarrow \) Reals. For each system, state whether it is linear (L), time-invariant (T), both (LTI), or neither (N).

(a) \( \forall n, \quad y(n) = x(-n). \) L

(b) \( \forall n, \quad y(n) = [x(n) + x(n - 1)]^2. \) TI
The $R, L, C$ circuit in the figure has for its input signal the voltage $x$ and its output signal is the inductor current $y$. From Kirchhoff’s law one can determine that these signals are related by the differential equation

$$\forall t, \quad RLC \frac{d^2 y(t)}{dt^2} + L \frac{dy(t)}{dt} + Ry(t) = x(t).$$

(a) 6 points Find the frequency response $H : \text{Reals} \rightarrow \text{Complex}$ of this system.

The frequency response is obtained by setting $\forall t, x(t) = e^{i\omega t}, y(t) = H(\omega) e^{i\omega t}$, substituting in the differential equation, to get

$$\forall \omega \in \text{Reals}, \quad H(\omega) = \frac{1}{R - RLC\omega^2 + i\omega}.$$ (b) 7 points Obtain an expression for the amplitude response and the phase response, assuming $R = L = C = 1$.

Substituting gives

$$H(\omega) = \frac{1}{(1 - \omega^2) + i\omega}.$$ (c) $\forall n, \quad y(n) = n[x(n) + x(n - 1)]$. L

(d) $\forall n, \quad y(n) = x(2n)$. L

(e) $\forall n, \quad y(n) = [x(n) + x(n + 1)]/2$. LTI
which in polar coordinates gives the amplitude response

\[ |H(\omega)| = \frac{1}{\sqrt{(1 - \omega^2)^2 + \omega^2}} \]

and the phase response

\[ \angle H(\omega) = -\tan^{-1} \frac{\omega}{1 - \omega^2} \]

(c) **7 points** Sketch the amplitude response and the phase response. Carefully mark the values for \( \omega = 0, 1 \) and \( \omega \rightarrow \infty \).

We have

\[ H(0) = 1, \quad H(1) = \frac{1}{i} = e^{-i\pi/2}, \quad \lim_{\omega \rightarrow \infty} |H(\omega)| = 0, \quad \lim_{\omega \rightarrow \infty} \angle H(\omega) = -\pi. \]

5. **15 points, 5 points each part** Fill in the blanks:

(a) The five roots of \( z^5 = 1 \) are:

\[ z = e^{2n\pi/5}, \quad n = 0, 1, 2, 3, 4. \]

(b) \( \forall t, \cos(\omega t) + \cos(\omega t + \pi/2) = ReAe^{i[\omega t + \phi]} \)

in which \( A = \sqrt{2}, \phi = \pi/4. \)

(c) The polar representation of the following numbers are:

\[ 1 + i = \sqrt{2}e^{i\pi/4} \]
\[ 1 - i = \sqrt{2}e^{-i\pi/4} \]
\[ [1 + i]^{-1} = \frac{1}{\sqrt{2}}e^{-i\pi/4} \]