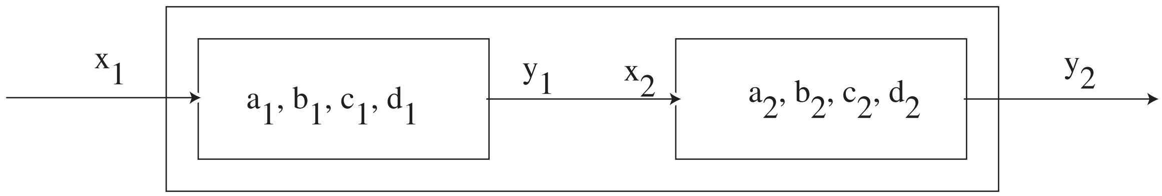
EECS20n, Quiz 4, 03/19/04, Solution

1. **10 points** Two linear systems are combined in a cascade composition as shown below:



The two systems, indexed $i = 1, 2$, are 1-dimensional with scalar input x_i, scalar output y_i, initial state $s_i(0)$ and update equations:

$$\begin{aligned} s_i(n+1) &= a_i s_i(n) + b_i x_i(n) \\ y_i(n) &= c_i s_i(n) + d_i x_i(n) \end{aligned}$$

The cascade composition means that $x_2 = y_1$. Write down the state, initial state, and update equations for the composite system.

Answer Observe that for all n

$$x_2(n) = y_1(n) = c_1 s_1(n) + d_1 x_1(n). \tag{1}$$

So using (1) twice,

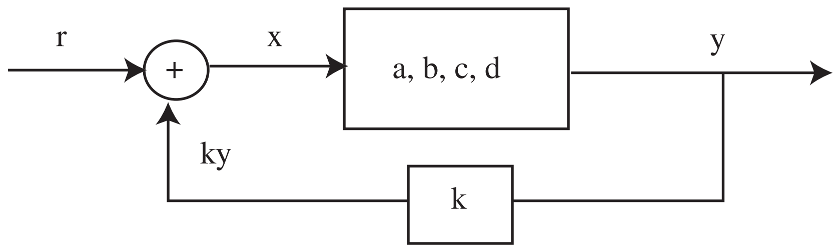
$$\begin{aligned} s_2(n+1) &= a_2 s_2(n) + b_2 x_2(n) = a_2 s_2(n) + b_2 c_1 s_1(n) + b_2 d_1 x_1(n) \\ y_2(n) &= c_2 s_2(n) + d_2 x_2(n) = c_2 s_2(n) + d_2 c_1 s_1(n) + d_2 d_1 x_1(n) \end{aligned}$$

Hence the update equation for the composite system is

$$\begin{aligned} \begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} &= \begin{bmatrix} a_1 & 0 \\ b_2 c_1 & a_2 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 d_1 \end{bmatrix} x_1(n) \\ y_2(n) &= \begin{bmatrix} d_2 c_1 & c_2 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + [d_2 d_1]\, x_1(n) \end{aligned}$$

Hence the state is $s(n) = [s_1(n), s_2(n)]^T \in R^2$; initial state is $s(0) = [s_1(0), s_2(0)]^T$; update equation is as above.

2. **10 points** A 1-dimensional system with scalar input x, scalar output y, state s, is put in feedback composition with input r and output y as shown below:



What are the state and the update equations for the feedback composition?

Answer The 'inner' system has update equation

$$\begin{aligned} s(n+1) &= as(n) + bx(n) \quad (2) \\ y(n) &= cs(n) + dx(n) \quad (3) \end{aligned}$$

The composite system has input $r(n)$ at time n. From (3),

$$x(n) = r(n) + ky(n) = r(n) + k[cs(n) + dx(n)]$$

which gives

$$\begin{aligned} x(n) &= \frac{1}{1-dk}r(n) + \frac{kc}{1-dk}s(n) \\ y(n) &= cs(n) + \frac{dkc}{1-dk}s(n) + \frac{d}{1-dk}r(n) \\ &= [c + \frac{dkc}{1-dk}]s(n) + \frac{d}{1-dk}r(n) \quad (4) \end{aligned}$$

Hence the composite system's update equation is:

$$\begin{aligned} s(n+1) &= [a + \frac{bkc}{1-dk}]s(n) + [\frac{b}{1-dk}]r(n) \\ y(n) &= [c + \frac{dkc}{1-dk}]s(n) + [\frac{d}{1-dk}]r(n) \end{aligned}$$

For the composition to be well-formed we must have $1 - dk \neq 0$. The state of the composition at time n is $s(n)$; its initial state is $s(0)$.