EECS 20n, Diagnostic Takehome Exam, 1/21/04, Solution

1. Let \( z_1 = 3 + 4i \) and \( z_2 = 5 + 12i \) be two complex numbers. Then

   (a) \( z_1 + z_2 = 8 + 16i \)

   (b) \( z_1 * z_2 = 15 + 48i^2 + 56i = -33 + 56i \)

   (c) \( z_2 / z_1 = [(5 + 12i)/(3 + 4i)] * [(3 - 4i)/(3 - 4i)] = (63 + 16i)/25 = 63/25 + 16/25i \)

2. (a) \( e^{ix} = \cos(x) + i\sin(x) = -1 \)

   (b) Show why \( \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \).

      Follows by repeatedly using the formulas (p. 626 of text),

      \[
      \begin{align*}
      \cos(a + b) &= \cos(a) \cos(b) - \sin(a) \sin(b) \\
      \sin(a + b) &= \sin(a) \cos(b) + \cos(a) \sin(b)
      \end{align*}
      \]

   (c) Express \( \sin 3\theta \) in terms of \( \sin \theta \).

      Using the same formulas, one gets

      \[
      \sin 3\theta = 3\sin \theta - 4\sin^3 \theta
      \]

(a) Does \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) converge? Why?

Use the integral test: the function

\[
\begin{align*}
    f : [2, \infty) \rightarrow \text{Reals,} & \quad f(t) = 1/(t - 1)^2,
\end{align*}
\]

satisfies, \( 1/n^2 \leq f(t) \), \( n \leq t \leq n + 1 \), so

\[
\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_{2}^{\infty} f(t) dt < \infty.
\]

(b) What is \( \lim_{x \to 0} \frac{\sin x}{x} = \)?

By L’Hopital’s rule (see p. 58 of text),

\[
\lim_{x \to 0} \frac{\sin x}{x} = \frac{d/dx \sin x(0)}{d/dx x(0)} = 1
\]

3. Solve the following first order linear differential equation:

\[
\frac{dy}{dx} = 2x + 1,
\]

with the initial condition \( y(0) = 0 \). What is \( y(1) \)? Plot \( y(x) \) for \( 0 \leq x \leq 1 \).

The solution is given by

\[
y(x) = \int_{0}^{x} [2s + 1] ds = x^2 + x
\]

So \( y(1) = 2 \). The plot is not shown: it is a quadratic function, starting at \( y(0) = 0 \).
4. Let $A$ be the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

(a) Verify $A^2 = A$.

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} = A$$

(b) Is it invertible? What is $A^{-1}$?

$A$ is **NOT** invertible, since $\det(A) = 0$, so $A^{-1}$ does not exist.

(c) Find all its eigenvalues.

The eigenvalues are the solutions of the characteristic equation,

$$\det(sI - A) = (s - 1)[(s - 0.5)^2 - 0.25] = s(s - 1)^2 = 0,$$

or $\{0, 1\}$: one eigenvalue at 0, and a double eigenvalue at 1.