EECS 20n, Diagnostic Takehome Exam, 1/21/04, Solution

1. Let $z_1 = 3 + 4i$ and $z_2 = 5 + 12i$ be two complex numbers. Then

   (a) $z_1 + z_2 = \boxed{8 + 16i}$

   (b) $z_1 * z_2 = 15 + 48i^2 + 56i = \boxed{-33 + 56i}$

   (c) $z_2/z_1 = [(5+12i)/(3+4i)] * [(3-4i)/(3-4i)] = (63+16i)/25 = \boxed{63/25 + 16/25i}$

2. (a) $e^{i\pi} = \cos(\pi) + i\sin(\pi) = \boxed{-1}$

   (b) Show why $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
   Follows by repeatedly using the formulas (p. 626 of text),

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(a+b) &= \sin(a)\cos 9b) + \cos(a)\sin(b) \end{aligned}$$

   (c) Express $\sin 3\theta$ in terms of $\sin\theta$.
   Using the same formulas, one gets

$$\sin 3\theta = \boxed{3\sin\theta - 4\sin^3\theta}$$

   (a) Does $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converge? Why?
   Use the integral test: the function

$$f : [2, \infty) \to \textit{Reals}, \quad f(t) = 1/(t-1)^2,$$

   satisfies, $1/n^2 \le f(t),\ n \le t \le n+1$, so

$$\boxed{\sum_{n=2}^{\infty} \frac{1}{n^2} \le \int_2^{\infty} f(t)dt < \infty.}$$

   (b) What is
   $\lim_{x\to 0} \frac{\sin x}{x} =$
   By L'Hopital's rule (see p. 58 of text),

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{d/dx\ \sin x(0)}{d/dx\ x(0)} = \boxed{1}$$

3. Solve the following first order linear differential equation:

$$\frac{dy}{dx} = 2x + 1,$$

   with the initial condition $y(0) = 0$. What is $y(1)$? Plot $y(x)$ for $0 \le x \le 1$.
   The solution is given by

$$y(x) = \int_0^x [2s + 1]ds = x^2 + x$$

   So $\boxed{y(1) = 2}$. The plot is not shown: it is a quadratic function, starting at $y(0) = 0$.

4. Let A be the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

(a) Verify $A^2 = A$.

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

(b) Is it invertible? What is A^{-1}?
A is $\boxed{\text{NOT}}$ invertible, since $\det(A) = 0$, so A^{-1} does not exist.

(c) Find all its eigenvalues.
The eigenvalues are the solutions of the characteristic equation,

$$\det[sI - A] = (s - 1)[(s - 0.5)^2 - 0.25] = s(s - 1)^2 = 0,$$

or $\boxed{\{0, 1\}}$: one eigenvalue at 0, and a double eigenvalue at 1.