Complex Numbers

Last time we
- Characterized linear discrete time systems by their impulse response
- Formulated zero input output response using impulse response
- Computed impulse response for example systems
- Saw examples of FIR and IIR systems

Today we will
- Review properties of complex numbers which we will use frequently as we continue to analyze linear systems

“Enrichment” exercises provided for those already fluent...

Imaginary Number

Can we always find roots for a polynomial? The equation
\[ x^2 + 1 = 0 \]
has no solution for \( x \) in the set of real numbers.
If we define a number that satisfies the equation
\[ x^2 = -1 \]
that is,
\[ x = \sqrt{-1} \]
then we can always find the \( n \) roots of a polynomial of degree \( n \).
We call the solution to the above equation the \textit{imaginary number}, also known as \( i \).
The imaginary number is often called \( j \) in electrical engineering.
Imaginary numbers ensure that all polynomials have roots.
Imaginary Arithmetic

Arithmetic with imaginary works as expected:

\[ i + i = 2i \]
\[ 3i - 4i = -i \]
\[ 5(3i) = 15i \]

To take the product of two imaginary numbers, remember that \( i^2 = -1 \):

\[ i \cdot i = -1 \]
\[ i^3 = i \cdot i^2 = -i \]
\[ i^4 = 1 \]
\[ 2i \cdot 7i = 14i \]

Dividing two imaginary numbers produces a real number:

\[ 6i \div 2i = 3 \]

Enrichment

Roughly speaking, the reason all our arithmetic works out as planned is because Complex is an algebraic structure called a field, and it works roughly the way the field Reals works. Verify the nine field axioms for all \( x, y \) and \( z \) in Complex:

1. Addition is commutative: \( x + y = y + x \)
2. Addition is associative: \( x + (y + z) = (x + y) + z \)
3. The Zero Property: There is a unique element 0 such that \( x + 0 = x \)
4. Additive Inverse Property: To each \( x \) there corresponds a unique element denoted \(-x\) with the property that \( x + (-x) = 0 \)
5. Multiplication is commutative: \( xy = yx \)
6. Multiplication is associative: \( x(yz) = (xy)z \)
7. The Multiplicative Identity Property: There is a unique non-zero element 1 such that \( x1 = x \)
8. Multiplicative Inverse Property: To each non-zero element \( x \) there corresponds a unique element denoted as \( x^{-1} \) such that \( x^{-1}x = 1 \)
9. The Distributive Property: \( x(y + z) = xy + xz \)

Complex Numbers

We define a complex number with the form

\[ z = x + iy \]

where \( x, y \) are real numbers.

The complex number \( z \) has a real part, \( x \), written \( \text{Re}(z) \).

The imaginary part of \( z \), written \( \text{Im}(z) \), is \( y \).

- Notice that, confusingly, the imaginary part is a real number.

So we may write \( z \) as

\[ z = \text{Re}(z) + i \text{Im}(z) \]
**Set of Complex Numbers**

The set of complex numbers, therefore, is defined by

\[ \text{Complex} = \{ x + iy \mid x \in \text{Reals}, y \in \text{Reals}, \text{and} \ i = \sqrt{-1} \} \]

Every real number is in Complex, because

\[ x = x + i0; \]

and every imaginary number \( iy \) is in Complex, because

\[ iy = 0 + iy. \]

**Equating Complex Numbers**

Two complex numbers

\[ z_1 = x_1 + iy_1 \]
\[ z_2 = x_2 + iy_2 \]

are equal if and only if their real parts are equal and their imaginary parts are equal.

That is, \( z_1 = z_2 \) if and only if

\[ \text{Re}\{z_1\} = \text{Re}\{z_2\} \]
and
\[ \text{Im}\{z_1\} = \text{Im}\{z_2\} \]

So, we really need two equations to equate two complex numbers.
**Complex Arithmetic**

In order to add two complex numbers, separately add the real parts and imaginary parts.

\[(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)\]

The product of two complex numbers works as expected if you remember that \(i^2 = -1\).

\[(1 + 2i)(2 + 3i) = \]
\[= \]
\[= \]

In general,

\[(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\]

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**Complex Conjugate**

The **complex conjugate** of \(x + iy\) defined to be \(x - iy\).

To take the conjugate, replace each \(i\) with \(-i\).

The complex conjugate of a complex number \(z\) is written \(z^*\).

Some useful properties of the conjugate are:

\[z + z^* = 2 \text{ Re}\{z\}\]
\[z - z^* = 2i \text{ Im}\{z\}\]
\[zz^* = \text{ Re}\{z\}^2 + \text{ Im}\{z\}^2\]

Notice that \(zz^*\) is a positive real number.

Its positive square root is called the **modulus** or **magnitude** of \(z\), and is written \(|z|\).

\[|z| = \sqrt{zz^*} = \sqrt{\text{ Re}\{z\}^2 + \text{ Im}\{z\}^2}\]
Dividing Complex Numbers

The way to divide two complex numbers is not as obvious. But, there is a procedure to follow:
1. Multiply both numerator and denominator by the complex conjugate of the denominator.
2. The denominator is now real; divide the real part and imaginary part of the numerator by the denominator.

\[
\frac{3 - 4i}{6 + 8i} =
\]

Complex Exponentials

The exponential of a real number \( x \) is defined by a series:

\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

Recall that sine and cosine have similar expansions:

\[
\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots
\]

\[
\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots
\]

We can use these expansions to define these functions for complex numbers.

Enrichment

For a complex number \( z \), what is \( \log z \)? See if you can define it. Find the complex numbers \( w \) such that \( z = e^w \). Hint: express \( z \) in polar form.
**Complex Exponentials**

Put an imaginary number $iy$ into the exponential series formula:

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots$$

$$e^{iy} =$$

Look at the real and imaginary parts of $e^{iy}$:

$$\text{Re}(e^{iy}) =$$

$$\text{Im}(e^{iy}) =$$

**Euler’s Formula**

This gives us the famous identity known as Euler’s formula:

$$e^{iy} = \cos(y) + i \sin(y)$$

From this, we get two more formulas:

$$\cos(y) =$$

$$\sin(y) =$$

Exponential functions are often easier to work with than sinusoids, so these formulas can be useful.

The following property of exponentials is still valid for complex $z$:

$$e^{Z_1 + Z_2} = e^{Z_1} e^{Z_2}$$

Using the formulas on this page, we can prove many common trigonometric identities. Proofs are presented in the text.
**Cartesian Coordinates**

The representation of a complex number as a sum of a real and imaginary number

\[ z = x + iy \]

is called its **Cartesian form**.

The Cartesian form is also referred to as **rectangular form**.

The name “Cartesian” suggests that we can represent a complex number by a point in the real plane, Reals².

We often do this, with the real part \( x \) representing the horizontal position, and the imaginary part \( y \) representing the vertical position.

The set Complex is even referred to as the “complex plane”.

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**Complex Plane**

![Complex Plane Diagram](image)
**Polar Coordinates**

In addition to the Cartesian form, a complex number $z$ may also be represented in **polar form**:

$$z = r \ e^{i\theta}$$

Here, $r$ is a real number representing the magnitude of $z$, and $\theta$ represents the angle of $z$ in the complex plane.

Multiplication and division of complex numbers is easier in polar form:

$$Z_1 \ Z_2 = \frac{Z_1}{Z_2}$$

Addition and subtraction of complex numbers is easier in Cartesian form.

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**Converting Between Forms**

To convert from the Cartesian form $z = x + iy$ to polar form, note: