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Continuous-Time Dynamic Linear Systems

Lecture 20:

EECS 20 N—March 5, 2001
• Announcements

  - Reading assignment: Chapter 6 of Lee and Varaiya
  - This week's lab due in two weeks only
  - No lab next week (3/12–3/16)
  - Still, go to lab sessions for problem set discussions
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announcements
• discrete-time MIMO systems
• continuous-time systems
From Lect 19:

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\[ N_{\text{preys}}(n+1) - N_{\text{preys}}(n) = -KN_{\text{predators}}(n) \]

where \( K > 0 \) are given constants

\[ N_{\text{predators}}(n+1) - N_{\text{predators}}(n) = LN_{\text{preys}}(n) \]
suppose

• we put (or kill) a number $x^n$ of preys in the ecosystem at the beginning of each season

• we also put (or kill) a number $x^{\text{predators}}$ of predators in the ecosystem at the beginning of each season

• we put (or kill) a number $x^{\text{preys}}$ of preys in the ecosystem at the beginning of each season

Multi-Input, Multi-Output system
MIMO system

State equations:

\[
\begin{align*}
\begin{bmatrix}
(u)_{\text{predators}} x \\
(u)_{\text{preys}} x
\end{bmatrix} + (u)s 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} + (u)s 
\begin{bmatrix}
1 & T \\
Y & T & -1 & 1
\end{bmatrix} = (I + u)s
\end{align*}
\]
above describes population at times $t = 0, 2T, 3T, 4T, ...$.

Let $N_{\text{preys}}(t)$ be the number of prey above its equilibrium value at time $t$. Likewise for predators $N_{\text{predators}}(t)$. Let $T = 3$ months is the length of one season.

With this new notation, our seasonal model can be written as:

\[
\begin{align*}
(\tau) N_{\text{preys}}^T &= (\tau) N_{\text{predators}} - (L + \tau) N_{\text{predators}} \\
(\tau) N_{\text{preys}}^Y - &\quad = (\tau) N_{\text{preys}} - (L + \tau) N_{\text{preys}}
\end{align*}
\]

seasonal model
period-to-period model could describe change from time \( t \) to \( t + \eta \), where \( \eta \) stands for an arbitrary period length (say, one month)
what happens if we look at instantaneous changes in the population?

\[
\begin{align*}
(\tau) N_{\text{preys}} & = \frac{\eta}{N} \left( N - (\eta + \varphi) N \right) \\
(\tau) N_{\text{predators}} & = \frac{\lambda}{N} \left( N - (\eta + \varphi) N \right)
\end{align*}
\]

where \( \eta > 0 \) and \( \lambda > 0 \) are independent of \( \tau \)

\[
\chi \cdot \eta = (\eta) I \quad \varphi \cdot \eta = (\eta) K
\]

we will assume both (\( \eta \))I and (\( \eta \))K are proportional to period length

period-to-period model
Fact: for a function $f : \text{Reals} \to \text{Reals}$, we define the derivative of $f$ at $t$ by the limit

$$\frac{\eta}{(\eta f - (\eta + \tau) f) \leftarrow \eta \text{ lim} \to p} = (\eta f \frac{\eta p}{p})$$

other notation: $(\eta) f', (\eta) f$
Instead of modelling change of population from one season to next, let $t \rightarrow 0$ in the period-to-period model.

**Continuous-time model**

\[
\begin{align*}
\frac{dN}{dt} &= \kappa N \frac{p}{p} \\
\frac{dN}{dt} &= \lambda N \frac{p}{p}
\end{align*}
\]
state-space representation

\[
(t) s \left[ \begin{array}{cc} 0 & -\lambda \\ \mu & 0 \end{array} \right] = \frac{\mu p}{s p}
\]

then

\[
\exists \ \text{Real}\ s \in \text{Real}\ s \left[ \begin{array}{c} (u)_{\text{predators}}^N \\ (u)_{\text{preys}}^N \end{array} \right] = (t) s
\]

again with
plot of zero-input response of continuous-time predator-prey model

- - - prey

--- predator
\[
\dot{s}(t) = As(t) + Bx(t), \quad t \geq 0,
\]
\[
y(t) = Cs(t) + Dx(t),
\]

where

- \( t \) is the (continuous) time
- \( s(t) \) is the state vector
- \( \dot{s}(t) \) is the derivative of \( s(t) \)
- \( A, B, C, D \) are constant matrices

### Model:

**Continuous-Time Models**
How can we simulate

\[
\begin{align*}
\dot{s}(t) &= A s(t) + B x(t), \quad t \geq 0, \\
\text{on a computer?}
\end{align*}
\]

\[
(\tau)x_B + (\tau)s_C = (\tau)y \\
(\tau)x_A + (\tau)s_B = (\tau)s
\]
discretization of continuous-time model

which (approximately) describes system at times $t = 0, 2h, 3h, \ldots$

\[
\begin{align*}
(\dot{t})xA + (\dot{t})sC &= (\dot{t})h \\
0 < t < h, \quad (\dot{t})xBh + (\dot{t})s(A \cdot y + I) &= (y + t)s
\end{align*}
\]

Obtain the discrete-time model

where $\eta$ is a (small) discretization period

\[
\frac{\eta}{(\dot{t})s - (y + t)s} \approx (\dot{t})s
\]

one approach is to approximate

discretization of continuous-time model
RC circuit:

\[ R \]
basis of circuit analysis & design

\[ \frac{d}{dt} I(t) + \frac{I(t)}{C} = 0 \]

where

- \( I(t) \) is intensity of current
- \( R \) is the resistance (a constant)
- \( C \) is the capacitance (a constant)

The equation above is the equation of an RC circuit.
state-space representation

Let $s(t) = \mathcal{L}(I)$, then:

$$s(t) RC \frac{dI}{dt} = \frac{f(t)}{s(t)}$$

be the state

$$(t)I = (t)s$$

Let
Newton’s Law: “acceleration proportional to external force”

\[ m \frac{\partial}{\partial t} \frac{\partial z}{\partial t} = (t) f \]

where

- \( f(t) \) is the external force
- \( z(t) \) is its position at time \( t \)
- \( m \) is the mass of a point
or, of more complicated systems:
example: Galileo's experiment

Let an object fall from a tower; what is its altitude as a function of time?

Newton's law: $\text{md}^2 \frac{dz}{dt} = -g \frac{d^2z}{dt^2}$

where $g$ is a constant (due to gravity)

$\text{if } \text{z is a function of time, then:}$

$\frac{dz}{dt} = \frac{\text{d}z}{\text{d}t}$

$\text{Example: Galileo's experiment}$
state-space representation

Let

\[
(s(t)) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}
\]

be the state of a second-order system

then:

\[
(\dot{f})f \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (\dot{s})s \begin{bmatrix} 0 & m \\ 1 & 0 \end{bmatrix} = \frac{fp}{sp}
\]
The derivative operator associates to a function its derivative. The inverse operation is called integration. The function such that
\[ F(t) - F(0) = \int_0^t f(\tau) \, d\tau \]
is such that
\[ \dot{F}(t) = f(t) \quad \forall t \]
informally, we write
\[ F = \int f \]
(does not specify "initial condition" on \( F \))
Integral representation

Consider continuous-time system

\[ S \int_{0}^{t} \left[ (\tau)x_\mathcal{D} + (\tau)s_\mathcal{C} \right] = (\tau)\bar{h} \]

\[ 0 \leq \tau \leq t, \quad (\tau)x_\mathcal{B} + (\tau)s_\mathcal{A} = (\tau)s \]

Rewrite above as

\[ (\tau)x_\mathcal{D} + (\tau)s_\mathcal{C} = (\tau)\bar{h} \]

\[ 0 \leq \tau \leq t, \quad (\tau)x_\mathcal{B} + (\tau)s_\mathcal{A} = (\tau)s \]

Integral representation

(does not specify initial condition)
\[ sdot = A \cdot s + B \cdot x \]

\[ y = C \cdot s + D \cdot x \]