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Frequency Response (II)

Lecture 25:

EECS 20 N—March 19, 2001
Outline

Reading assignment: Chapter 8 of Lee and Varaiya

1. Composition of linear, time-invariant systems
2. Frequency response and impulse response
3. Frequency response of discrete-time systems
Consider the discrete-time exponential input $x(n) = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$ acting on an LTI (linear, time-invariant) discrete-time system denoted $S$. What is the zero-state output response?
assume the system $S$ is the delay operator $D_N$, defined by

where the scaling factor $H(\omega) = e^{-i\omega N}$

so, for complex exponential inputs $x_{\omega}(n) = e^{i\omega n}$,

we have

\[
D_N(x_{\omega})(n) = x_{\omega}(n - N) = e^{-i\omega N} x_{\omega}(n)
\]

\[\forall x \in \text{Ints} \rightarrow \text{Comps}, \forall n \in \text{Ints}, D_N(x)(n) = x(n - N)\]
\[(N - t)^{m}x(0)(^{m}xS) = (0)(^{m}xS)_{N^{m}t^{-0}} = (N)(^{m}xS) \quad \text{for } I \in N \land \exists \quad \text{apply this for } u = 0, \text{ and change in } N - N \quad \text{get}
\]
\[(u)(^{m}xS)_{N^{m}t^{-0}} = (N - u)(^{m}xS) \quad \text{for } I \in u, N \land \exists \quad \text{equating the two, we obtain:}
\]
\[(N - t)(^{m}xS) = (u)((^{m}xS)^{N}D)
\]
but, by definition of delay:
\[
(^{m}xS)_{N^{m}t^{-0}} = (^{m}x_{N^{m}t^{-0}})S = ((^{m}x)^{N}D)S
\]
from time-invariance and linearity of \(S\):
now, if \(S\) is a general discrete-time LTI system:

Response of arbitrary LTI systems
It is a complex number

the function \( \omega \rightarrow H(\omega) \) is called the frequency response of \( S \)

\[
(u)^{\omega}x(\omega)H = (u)^{\omega}(x_{\omega}) \quad \text{for } I \subseteq u \land
\]

then:

\[
(0)^{\omega}(x_{\omega}) = (\omega)H
\]

define

\[
u_{\omega}^\omega = (\omega)^{\omega}x
\]

Summary: Let \( (x_{\omega}) \) be the zero-state output response of a continuous-time LTI system \( S \) to a complex exponential input:

Frequency response for discrete-time systems
\[(\omega)H = (\omega + m)H \quad \forall m \in \mathbb{Z}\]

Thus, for discrete-time systems the frequency response is periodic with period \(2\pi\):

\[x(n) = e^{in(\omega + 2\pi K)} = (u_n)x\]

is the same as the discrete complex exponential with frequency \(\omega + 2\pi K\): for every \(K \in \mathbb{Z}\), a discrete complex exponential with frequency \(\omega\):
example: moving average

length-two moving average:

formally:

\[ y(n) = \frac{1}{2} \left( x(n) + x(n-1) \right) \]

by linearity, the frequencies response is the average of those of \( D_0 \), \( D_1 \):

\[ H(\omega) = \frac{1}{2} \left( 1 + e^{-i\omega} \right) \]
The system is a low-pass filter attenuated by the system as frequency increases: Higher frequencies are reduced.

Since $H(0) = 1$, a constant is transmitted without any amplitude reduction.

Bode magnitude plot of moving average
Relationship between impulse and frequency responses?

- Impulse response describes (zero-state) response to arbitrary signals.
- Frequency response describes (zero-state) response to arbitrary periodic signals.

For a linear, time-invariant (LTI) system, the relationship between impulse and frequency responses is:

\[ H(f) = \mathcal{F}\{h(t)\} \]

where \( H(f) \) is the frequency response and \( h(t) \) is the impulse response.
Impulse response is response to unit impulse at $n = 0$.

\[
(\delta - u)x(\delta)y \quad \text{in } \sum_{\infty}^{\infty} = (\delta)x(\delta - u)y \quad \text{in } \sum_{\infty}^{\infty} = (u)(x*y) = (u)y
\]

Find zero-state output response via convolution: •

Where $\delta (\delta - u)$ is unit impulse at time $k$.

\[
(\delta - u)\delta(\delta)x \quad \text{in } \sum_{\infty}^{\infty} = (u)x
\]

Decompose input signal as weighted sum of impulses: •

Impulse response describes (zero-state) response to arbitrary signals:
Interpretation

• The impulse response tells us what is the relative contribution of past inputs in the present output.

• LTl systems such that the sequence \( h(n) \) vanishes outside a window of finite length are called finite impulse response (FIR) filters.

• We will see later that we can design LTl systems to increase or decrease these contributions (this is called filtering).

• Interpretation
\[
\sum_{i=0}^{n-3} \frac{1}{3} \delta(\pi) + \frac{1}{3} \delta(\pi+1) + \frac{1}{3} \delta(\pi+2) + (\pi - u g(0) + (1 - u g(0)) + (u g(\pi))
\]
Frequency response describes (zero-state) response to arbitrary periodic signals:

- Decompose input periodic signal as weighted sum of pure exponentials via Fourier expansion:

$$x(n) = \sum_{k=0}^{p-1} X_k e^{i k \omega_0 n}$$

- Find zero-state output response by multiplying each component of frequency response $H(k\omega_0)$ by the fundamental frequency $\omega_0$ where $\omega_0 = \frac{2\pi}{d}$ is the fundamental frequency:

$$y(n) = \sum_{k=0}^{p-1} H(k\omega_0) X_k e^{i k \omega_0 n}$$
There are no frequency components in the output that were not in the input: the output consists of the same frequency components as the input, but with each component individually scaled.

LTl systems can be used to enhance or suppress certain frequency components (this is called filtering). The frequency response function $H(m)(\omega)$ characterizes which frequencies are enhanced or suppressed and also what phase shifts might be imposed on individual components by the system.
The relationship between impulse and frequency response since frequency response treats only periodic signals, and impulse response handles arbitrary ones, there must be a relationship between frequency response and impulse response.

Let's start from the zero-state output response to an arbitrary input:

\[
(y(n)) = \sum_{m=-\infty}^{\infty} h(m)x(n-m)
\]

where \( m \) is given, and applying this to a complex exponential input:

\[
x(n) = e^{i\omega n}
\]

\[
y(n) = \sum_{m=-\infty}^{\infty} h(m)e^{i\omega(n-m)} = (u)\tilde{y}
\]

Since frequency response treats only periodic signals, and impulse response
Frequency response uniquely determines the impulse response. We'll see later that the relationship can be inverted, that is, the discrete-time Fourier transform (DTFT) of the impulse response is the frequency response $H$. We say that

$$
\omega \in \mathbb{R} \quad \sum_{m=-\infty}^{\infty} \mathcal{F}(\omega) h = (\omega) H
$$

so the relationship is

$$
(\omega)x(\omega)H = (\omega)h
$$

On the other hand, we know that for periodic inputs,
Let's find the frequency and impulse responses of system $S^1 + S^2$

Consider the parallel connection between $S^1$ and $S^2$:

Parallel connection of two LTI systems.
parallel connection: results

For the parallel connection of two LTI systems $S_1$ and $S_2$, the following equations hold:

1. The impulse response of $S$ is the sum of the impulse responses $h$: $\mathcal{Z}_2 h + \mathcal{Z}_1 h = h$
2. The frequency response of $S$ is the sum of the frequency responses $H$: $\mathcal{Z}_2 H + \mathcal{Z}_1 H = H$
3. For the parallel connection of two LTI systems $S_1$ and $S_2$, $S = S_1 + S_2$. 

parallel connection: results
Let's find the frequency and impulse responses of system $S_1 \circ S_2$.

Consider the cascade connection between $S_1$ and $S_2$:

Cascaded connection of two LTI systems.
First, we note that $S^1 \circ S^2$ is LTI:

\[(\bar{S} \circ S^1) \circ D = (\bar{S} \circ D) \circ S^1 = D \circ (\bar{S} \circ S^1)\]

Therefore

\[x(n) \bar{H}(n) \bar{H} = (x(n) \bar{H}) \bar{H} = (x)(\bar{S} \circ S)\]

next, if $x(n) = e^{j\omega n}$, then

\[\forall \omega \in \text{Reals}, H(\omega) = H_1(\omega) H_2(\omega)\]
impulse response of cascade connection

if $S = S_1 \circ S_2$, then impulse response $h$ of $S$ is response to an impulse at $n = 0$:

$$h(n) = (S_1 \circ S_2)(\delta) = S_1(S_2(\delta))$$

by definition, $h_2$ is the impulse response of $S_2$, so

$$((\varphi(S_2))^\dagger S = (\varphi)((S_2) \circ S_1)^\dagger) = h$$

hence:

$$\mathcal{Z}(\varphi) = h$$

Input (\varphi) is the convolution of impulse response $h_1$ of $S_1$ with the output, \( h_2 \) is the impulse response of $S_2$ to an arbitrary input $\mathcal{Z}(\varphi) = h$.

Impulse response of cascade connection
The frequency response of \( S \) is the convolution of the frequency response \( H_1 \cdot H_2 \).

The impulse response of \( S \) is the convolution of the impulse response \( \eta \cdot \eta \).

For the cascade connection of two LTI systems \( S_1 \circ S_2 \), we'll see later that the Fourier transform of a convolution is indeed the product of the Fourier transforms.
Frequency and impulse responses are related by the discrete-time Fourier transform (DTFT). Frequency and impulse responses are related by the discrete-time Fourier transform (DTFT).