Fourier Transforms (III)

Lecture 32:

EECS 20 N—April 16, 2001
today's lecture:

Continuous-time and discrete-time Fourier transforms (CTFT and DTFT)

Properties and examples

Reading assignment: Chapter 9 of Lee and Varaiya
From last time, we've seen the following Fourier transforms:

- The Fourier series (FS) for a $p$-periodic continuous-time signal:

  $$ x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot \frac{1}{p} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} x(s) e^{-ik\omega_0 s} ds $$

where $\omega_0 = \frac{2\pi}{p}$.

- The inverse Fourier transform (IFT) for a $p$-periodic continuous-time signal:

  $$ X(k) = \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} x(t) e^{ik\omega_0 t} dt $$

Last time, we've seen the following Fourier transforms:
its discrete-time counterpart, the discrete Fourier series:

\[ u_0 \sum_{k=0}^{\gamma} X_{\mathbb{I}-d} = (u)x \]

and its inverse:

\[ u_0 \sum_{k=0}^{u} (u)x_{\mathbb{I}-d} = \gamma X \]

\( u \) of a periodic discrete-time signal \( x \).
Continuous-time Fourier transform

It remains to see a transform relating time and frequency for non-necessarily periodic, continuous-time signals: the

Continuous-time Fourier transform

\[ X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \]

and its inverse,

\[ x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \]

Periodic (discrete-time) signals

The discrete-time Fourier transform of a (non-necessarily periodic) signal

\[ X[k] = \sum_{n=-\infty}^{\infty} x[n] e^{-i2\pi nk/N} \]

and its inverse,

\[ x[n] = \frac{1}{N} \sum_{k=-N/2}^{N/2} X[k] e^{i2\pi nk/N} \]
Continuous-time Fourier transform

The frequency response and impulse response of a continuous-time LTI system are related by the continuous-time Fourier transform (CTFT)

\[ m \mathcal{F}_m \mathcal{F}(\omega)H \int_{-\infty}^{\infty} \frac{\mathcal{F}_m \mathcal{F}(\omega)H}{\pi} = (\hat{f}(t)) \forall t \in \text{Reals}, \forall \omega \in \text{Reals} \]

The inverse relation is (you could guess it)

\[ \mathcal{F}_m \mathcal{F}^{-1}(\omega)H \int_{-\infty}^{\infty} (\hat{f}(t)) = (m)H \forall m \in \text{Reals}, \forall \omega \in \text{Reals} \]

(proving this is more difficult)
CTFT can be viewed as a generalization of both the FS and DTFT where neither frequency nor time is discrete. The last remaining Fourier transform, where neither frequency nor time domain needs to be periodic, is the CTFT.
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**Summary**
\[(m)X(m)H = (m)X \quad \text{m A}
\]

In the above we recognize the inverse DFT of \((m)X(m)H\) as
\[
\mathcal{M} \eta \mathfrak{m} \{ (m)X(m)H \} = \int_{-\infty}^{0} \frac{\nu \mathcal{Z}}{1} = (u)\mathfrak{h}
\]

From linearity, obtain response to arbitrary input \(x\) as a weighted sum of exponential signals \(e^{j\omega n}\).

\[
\mathcal{M} \eta \mathfrak{m} \{ (m)X \} = \int_{-\infty}^{0} \frac{\nu \mathcal{Z}}{1} = (u)x
\]

Thus, from inverse DFT, let \(x\) be an input to a discrete-time system with frequency response.
In the time-domain, the output $y(n)$ to a discrete-time, LTI system is related to the input $x$ by convolution:

$$y(n) = (h * x)(n)$$

where $h$ is the impulse response and $x, y$ are the DTFT's of $H, X, Y$.

Moreover, convolution in the time-domain is equivalent to multiplication in the frequency domain:

$$X(m)H = XH = Y$$

where $X, Y$ are the DTFT's of $x$ and $y$ respectively. The previous page shows that, in the frequency domain, input and output are related by a simple multiplication:

$$H * X = Y$$

Thus, convolution in time-domain is equivalent to multiplication in frequency domain.

**Summary**
For a proof, see below:

\[ \mathcal{z}H_1 H = H \]

Response of \( S \) is the impulse response of \( S_1 \) \( S_2 \), then the impulse response of \( S \) is \( h = h_1 * h_2 \)

Frequency response of \( S \) is the frequency responses of \( S_1 \) \( S_2 \), then the frequency response of \( S \) is \( H = H_1 H_2 \)

Thus, for a cascade system, \( S = S_2 \circ S_1 \)
consider the feedback connection:

feedback connection

This system is LTI (since $S_1, S_2$ are). Frequency response of system?
the output is related to the input by

\[
Y(\omega) = H(\omega)X(\omega)
\]

where

\[
H(\omega) = H_1(\omega) - H_1(\omega)H_2(\omega)
\]

is the frequency response of the feedback system.

\[
\frac{\mathcal{Z}(\omega)H(\omega)\mathcal{H} - I}{\mathcal{H}} = (\omega)H
\]

this can be written as

\[
(\omega)X(\omega)\mathcal{H} = (\omega)\mathcal{H} \mathcal{Z}(\omega)H(\omega)\mathcal{H} - (\omega)\mathcal{H}
\]

from which we obtain:

\[
(x * \mathcal{H}) = ((\mathcal{H} * \mathcal{H}) * \mathcal{H}) - \mathcal{H}
\]

or

\[
(x)\mathcal{S} = ((\mathcal{H})\mathcal{S})\mathcal{S} - \mathcal{H}
\]
(Frequency or time) is real. Conjugate symmetric, then the function in the other domain is more generally, if a function in one domain (time or frequency) is

\[ X(f) = X(-f) \]

satisfy the conjugate symmetry property:

For example, the Fourier series coefficients of a real-valued, p-periodic

Continuous-time signal time-domain function is real the Fourier transforms are conjugate symmetric if the

Conjugate Symmetry
Let $x$ be a continuous-time signal with CFT $X$, what is the CFT $Y$ of the delayed signal $y(t) = x(t - \tau)$ for all $t \in \mathbb{R}$? Since the delay operator has frequency response $e^{-i\omega \tau}$, get

$$Y(\omega) = e^{-i\omega \tau} X(\omega).$$
• impulse function: assume \( x(t) = \delta(t) \) (the Dirac delta function), then the CFT is \( X(\omega) = 1 \).

**Proof:** Let \( \mathcal{S} \) be the system with impulse response \( \varrho \), and remember that \( \mathcal{S} \) satisfies \( (z * \varrho) = (z)^{\varrho} \) for every \( z \) in the frequency domain. Get \( \mathcal{S} \) for every \( \mathcal{S} \).

\[ Z = (m)Z(m)X \]

\[ I = (m)X \]

• delayed impulse: assume \( x(t) = \delta(t - \tau) \) (the Dirac delta function), then the CFT is \( X(\omega) = e^{-i\omega\tau} \).

**Proof:** \( \delta(t - \tau) \) is the impulse response of the delay operator.
\( x(n) = \delta(n+1) + \delta(n-1) \), where \( \delta \) is the Kronecker delta function.

The DTFT is real, since the time-domain function is conjugate symmetric and the DTFT is (by linearity of DFT)

\[
(m) \cos \omega = m_{-\omega} + m_{\omega} = (m)X
\]

\( \text{DFCT} \) is (by linearity of DFCT)

where \( \omega \) is the Kronecker delta function

\[
(I - u)\varphi + (I + u)\varphi = (u)x
\]

assume
constantsignals

\[ Y = (t)x \]

Thus, \( Y \) is the CTFT of the constant continuous-time signal \( x(t) \).

\[
(\mathcal{F})x = Y = \mathcal{F}_{\mathcal{T}} \mathcal{F}_{\mathcal{F}} \left( \mathcal{F} \right) x(t) = \mathcal{F}_{\mathcal{T}} \mathcal{F}_{\mathcal{F}} \left( \mathcal{F} \right) x(t)
\]

Exchanging the role of time and frequency in the above:

\[
1 = \mathcal{F}_{\mathcal{T}} \mathcal{F}_{\mathcal{F}} \left( \mathcal{F} \right) x(t) \mathcal{F}_{\mathcal{F}} \mathcal{F}_{\mathcal{T}}
\]

Remember that the CTFT of the Dirac \( \delta \) function is \( \mathcal{F} \delta(\omega) = 1 \), that is,

where \( Y \in \mathbb{R} \) is constant, CTFT

\[
Y = (t)x \quad \forall t
\]

Assume \( x(t) \) is constant signals

\[ \text{constant signals} \]
Let $x$ be a constant discrete-time signal,

$$x[n] = K \delta[n].$$

Remember the DFT is $2\pi$-periodic, so we study first the case when

$$\omega \in \mathbb{R}.$$